Practice exam 1.1

1. $\int_0^{\pi/8} \cos^2 x \cdot \sin^2 x \overset{①}{=} \int_0^{\pi/8} \frac{1}{4}\sin^2 2x\, dx \overset{②}{=} \int_0^{\pi/8} \frac{1}{8}(1-\cos 4x)\,dx =$

$= \frac{1}{8}\left(\frac{\pi}{8} - \frac{1}{4}\sin 4x\Big|_0^{\pi/8}\right) = \frac{1}{8}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \underline{\frac{1}{64}(\pi - 2)}$

① Use formula $\sin 2x = 2\sin x \cdot \cos x$.

② $\sin^2 2x = \frac{1}{2}(1-\cos 4x)$.

2. $f(x) = x^3 + 2x - 1$. Find $(f^{-1})'(2)$.
Let $f^{-1}(2) = x$; then $f(x) = 2 \Rightarrow x = 1$. $f'(x) = 3x^2 + 2$; $f'(1) = 5$
$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \underline{\frac{1}{5}}$.

3. Let $y = x^{a^x}$ $a > 0, x > 0$. Find y'.
$y = x^{a^x} = e^{a^x \cdot \ln x}$; $y' = e^{a^x \ln x} \cdot (a^x \cdot \ln x)' = e^{a^x \cdot \ln x}(a^x \cdot \ln a \cdot \ln x + \frac{a^x}{x}) =$
$= \underline{x^{a^x} \cdot a^x \cdot (\ln a \cdot \ln x + \frac{1}{x})}$
or consider $\ln y = a^x \cdot \ln x$,
$\frac{y'}{y} = (\ln y)' = (a^x \cdot \ln x)' = a^x(\ln a \cdot \ln x + \frac{1}{x})$, so
$y' = y \cdot a^x(\ln a \cdot \ln x + \frac{1}{x}) = \underline{x^{a^x} \cdot a^x(\ln a \ln x + \frac{1}{x})}$

4. Solve for x $2^x = 3^{1-x}$.
$\ln 2^x = \ln 3^{(1-x)} \Rightarrow x \cdot \ln 2 = (1-x) \cdot \ln 3 \Rightarrow x(\ln 2 + \ln 3) = \ln 3 \Rightarrow$
$\underline{x = \frac{\ln 3}{\ln 6}}$.

5. $\int \frac{6}{9+x^2}\,dx = \int \frac{6}{9(1+(\frac{x}{3})^2)}\,dx = \left[\begin{array}{l} u = x/3 \\ du = \frac{1}{3}dx \\ 3du = dx \end{array}\right] = \frac{6 \cdot 3}{9}\int \frac{1}{1+u^2}\,du =$

$= 2\tanh^{-1} u + C = 2\tanh^{-1}\frac{x}{3} + C$.

$\int \frac{6}{9-x^2}\,dx = \int \left(\frac{1}{3-x} + \frac{1}{3+x}\right)dx = -\ln|3-x| + \ln|3+x| + C = \ln\left|\frac{3+x}{3-x}\right| + C$.

6. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx = \left[\begin{array}{l} u = \sin^{-1}(x) \\ du = \frac{dx}{\sqrt{1-x^2}} \end{array}\right] = \int_{\sin^{-1} 0}^{\sin^{-1}(1/\sqrt{2})} u \cdot du = \frac{1}{2}u^2\Big|_0^{\pi/4} = \frac{\pi^2}{32}$

7. $\lim_{x\to\pi} \frac{\sin x}{\ln(\pi/x)} = \left[\frac{0}{0}\right] = \lim_{x\to\pi} \frac{\cos x}{-\frac{\pi}{x^2}\cdot\frac{x}{\pi}} = \frac{-\cos\pi}{\frac{1}{\pi}} = \pi$.