THE STRUCTURE OF THE v_2-LOCAL ALGEBRAIC tmf RESOLUTION

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ABSTRACT. We give a complete description of the E_1-term of the v_2-local as well as g-local algebraic tmf resolution.

CONTENTS

1. INTRODUCTION

Let bo denote the connective real K-theory spectrum. Mahowald and his collaborators used the bo resolution (aka the bo-based Adams spectral sequence) to study stable homotopy groups to great effect. Specifically, they computed the image of the J-homomorphism [DM89], proved the 2-primary height 1 telescope conjecture [Mah81], [LM87], computed the unstable v_1-periodic homotopy groups of spheres [Mah82], and applied homotopy theoretic methods to a variety of geometric problems [DGM81].

The spectrum bo has two distinct advantages that lend itself to these applications at the prime 2. Firstly, $\pi_0\mathrm{bo}$ is torsion free and $\pi_*\mathrm{bo}$ is Bott periodic (i.e. v_1-torsion free), so it is equipped to detect the zeroth and first layers of the chromatic filtration. Secondly, v_1-periodic homotopy at the prime 2 is more complicated than at odd primes, and this is witnessed by the elements η and η^2 generating additional anomalous torsion [Ada66]. These elements and their v_1-multiples are detected by the bo-Hurewicz homomorphism

$$\pi_*^s \to \pi_*\mathrm{bo}.$$

At chromatic height 2, the 2-primary stable stems have a vast collection of anomalous torsion, and a significant portion of this v_2-periodic torsion is detected by the spectrum tmf of topological modular forms (see [BMQ21]). As such the tmf resolution represents a significant upgrade to the bo resolution. Indeed, partial analysis of the tmf resolution has resulted in numerous powerful results [BHHM08], [BHHM20], [BBB+21], [BMQ21].

For a spectrum X, the *tmf resolution* of X is the tower of cofiber sequences

$$\begin{tikzcd} X \arrow[d] & \Sigma^{-1}\overline{\mathrm{tmf}} \wedge X \arrow[l] \arrow[d] & \Sigma^{-2}\overline{\mathrm{tmf}}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \mathrm{tmf} \wedge X & \Sigma^{-1}\mathrm{tmf} \wedge \overline{\mathrm{tmf}} \wedge X & \Sigma^{-2}\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^{\wedge 2} \wedge X & \end{tikzcd} \tag{1.1}$$

Here $\overline{\mathrm{tmf}}$ is the cofiber of the unit

$$S \to \mathrm{tmf} \to \overline{\mathrm{tmf}}.$$

Applying π_* to the tower above results in the tmf*-based Adams spectral sequence*

$${}^{\mathrm{tmf}}E_1^{n,t}(X) = \pi_t(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^{\wedge n} \wedge X) \Rightarrow \pi_{t-n}X.$$

Ultimately, the successful applications of the tmf-resolution so far have been limited by our ability to compute the E_1-page of the tmf-based Adams spectral sequence — computations to date have relied on computations of the E_1-page in certain regions. Unlike the bo case, we are not able to completely compute this E_1 page for $X = S$. The goal of this paper is to make a significant step towards rectifying this deficiency.

The computations of the E_1-page that have been successfully performed used the classical Adams spectral sequence. We focus our attention at the prime 2. Recall that for a connective spectrum Y, the *mod* 2 *Adams spectral sequence* (ASS) takes the form

$${}^{ass}E_2^{s,t}(Y) = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, H_*Y) \Rightarrow \pi_{t-s}Y_2^\wedge$$

where H_* denotes mod 2 homology and A_* is the dual Steenrod algebra. The E_1-term of the tmf-resolution than can then itself be approached by computing the ASS's

$${}^{ass}E_2^{s,t}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n \wedge X) \Rightarrow \pi_{t-s}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n \wedge X) = {}^{\mathrm{tmf}}E_1^{n,t-s}(X).$$

In practice, given the computation of the E_2-pages, these Adams spectral sequences can be completely computed, as the majority of the differentials can be deduced from the Adams spectral sequence for tmf (as computed in [BR22]). The tmf-resolution can then be studied through the Miller square [Mil81]

$$\begin{tikzcd}
{}^{ass}E_2^{s,t}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n \wedge X) \arrow[r, Rightarrow, "ASS"] \arrow[d, Rightarrow, "\text{alg tmf res}"'] & {}^{\mathrm{tmf}}E_1^{n,t-s}(X) \arrow[d, Rightarrow, "\text{tmf res}"] \\
{}^{ass}E_2^{s+n,t}(X) \arrow[r, Rightarrow, "ASS"'] & \pi_{t-s-n}X_2^\wedge
\end{tikzcd}$$

Here, the left side of the square is the *algebraic* tmf*-resolution*, the analog of the tmf-resolution obtained by applying Ext_{A_*} to (1.1). The starting point is therefore the computation of the E_1-page of the algebraic tmf resolution of the sphere

$${}^{ass}E_2^{s,t}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n).$$

Analogous to the case of the bo-resolution and the $BP\langle 2\rangle$-resolution [Mah81] [Cul19], we propose the following conjecture.

Conjecture 1.2. *The map*

$${}^{ass}E_2^{s,t}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n) \to v_2^{-1}\,{}^{ass}E_2^{s,t}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n)$$

is injective for $s > 0$.

This conjecture is consistent with computations in low degrees (see, for instance, [BOSS19]). It implies a good-evil decomposition of the tmf-resolution of the sphere, analogous to that of [BBB+20], [BBB+21].

In this paper we give a complete computation of

$$v_2^{-1}\,{}^{ass}E_2^{*,*}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n).$$

We now summarize the main results.

For a graded Hopf algebra Γ over k, let $\mathcal{D}_\Gamma$ denote Hovey's stable homotopy category of Γ-comodules. Briefly, $\mathcal{D}_\Gamma$ is similar to the derived category, with the chief difference that weak equivalences are defined to be the $\pi^\Gamma_{*,*}$-isomorphisms, where for a Γ-comodule M, the homotopy groups $\pi^\Gamma_{*,*}$ are defined to be

$$\pi^\Gamma_{n,s}(M) := \mathrm{Ext}^{s,s+n}_\Gamma(k, M).$$

For $M \in \mathcal{D}_\Gamma$, we let $\Sigma^{n,s}M$ denote a shift in internal degree by $s+n$ and in cohomological degree by s, so we have

$$\pi^\Gamma_{k,l}(\Sigma^{n,s}M) = \pi^\Gamma_{k-n,l-s}(M)$$

and

$$[\Sigma^{n,s}k, M]_\Gamma = \pi^\Gamma_{n,s}(M).$$

For a spectrum X, we shall let

$$\underline{X} \in \mathcal{D}_{A_*}$$

denote the object associated to the mod 2 homology H_*X. In this notation the ASS takes the form

$${}^{ass}E_2^{s,t}(X) = \pi^{A_*}_{t-s,s}(\underline{X}) \Rightarrow \pi_{t-s}X_2^\wedge.$$

Since $\underline{\mathrm{tmf}} = (A /\!\!/ A(2))_*$ [Mat16] (where $A(2)$ is the subalgebra of the mod 2 Steenrod algebra generated by Sq^1, Sq^2, and Sq^4), we have a change of rings isomorphism

$$\pi^{A_*}_{*,*}(\underline{\mathrm{tmf}} \otimes M) \cong \pi^{A(2)_*}_{*,*}(M) \tag{1.3}$$

for any $M \in \mathcal{D}_{A_*}$. Therefore the E_1-term of the algebraic tmf-resolution takes the form

$$^{ass}E_2^{*,*}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^{\wedge n}) \cong \pi^{A(2)_*}_{*,*}(\overline{\underline{\mathrm{tmf}}}^{\otimes n}).$$

There is a decomposition [BHHM08]

$$\overline{\underline{\mathrm{tmf}}}^{\otimes n} \simeq \bigoplus_{i_1,\dots,i_n>0} \Sigma^{8(i_1+\cdots+i_n)}\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \tag{1.4}$$

in $\mathcal{D}_{A(2)_*}$, where $\underline{\mathrm{bo}}_i$ denotes the homology of the ith bo-Brown-Gitler spectrum (see Section 2).

For an object $M \in \mathcal{D}_{A(2)_*}$, the localization $v_2^{-1}M$ denotes the localization of M with respect to the element

$$v_2^8 \in \pi^{A(2)_*}_{48,8}(\mathbb{F}_2),$$

so we have

$$v_2^{-1}\, {}^{ass}E_2^{*,*}(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^{\wedge n}) \cong \pi^{A(2)_*}_{*,*}(v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}).$$

We will prove

Theorem 1.5 (see Corollary 8.6 and (2.9))**.** *There are equivalences in $\mathcal{D}_{A(2)_*}$*

$$v_2^{-1}\underline{\mathrm{bo}}_{2j} \simeq \Sigma^{8j} v_2^{-1}\underline{\mathrm{bo}}_j \oplus \Sigma^{8j+8,1} v_2^{-1}\underline{\mathrm{bo}}_{j-1},$$
$$v_2^{-1}\underline{\mathrm{bo}}_{2j+1} \simeq v_2^{-1}\Sigma^{8j}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1.$$

The splittings of (1.4) and Theorem 1.5 inductively imply that in $\mathcal{D}_{A(2)_*}$ the objects $v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}$ split as a wedge of bigraded suspensions of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$. We are left with identifying these explicitly.

To this end we will introduce an object

$$\underline{\mathrm{TMF}_0(3)} \in \mathcal{D}_{A(2)_*}$$

which serves as an algebraic version of the tmf-module $\mathrm{TMF}_0(3)$ (the theory of topological modular forms associated to the congruence subgroup $\Gamma_0(3) < SL_2(\mathbb{Z})$), and prove

Theorem 1.6 (Proposition 5.1 and 5.2)**.** *There are splittings in $\mathcal{D}_{A(2)_*}$*

$$v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 3} \simeq 2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1 \oplus \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)},$$
$$\underline{\mathrm{TMF}_0(3)} \otimes \underline{\mathrm{bo}}_1 \simeq \Sigma^{24,3}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{40,6}\underline{\mathrm{TMF}_0(3)}.$$

The splittings of Theorem 1.6 imply that the objects $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$ split in $\mathcal{D}_{A(2)_*}$ as a direct sum of bigraded suspensions of copies of $v_2^{-1}\mathbb{F}_2$, $v_2^{-1}\underline{\mathrm{bo}}_1$, $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}$, and $\underline{\mathrm{TMF}_0(3)}$.

Putting this all together, we have the following theorem (see Corollary 8.7 for a more precise formulation).

Theorem. *There is a splitting of*

$$v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n} \in \mathcal{D}_{A(2)_*}$$

into a well-described sum of various bigraded suspensions of

- $v_2^{-1}\mathbb{F}_2$,
- $v_2^{-1}\underline{\mathrm{bo}}_1$,
- $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}$,
- $\underline{\mathrm{TMF}_0(3)}$.

The most subtle step to all of this is the first equivalence of Theorem 1.5. Indeed an explicit exact sequence (see (2.5)) of [BHHM08] implies that $v_2^{-1}\underline{\mathrm{bo}}_{2j}$ is built from $v_2^{-1}\Sigma^{8j}\underline{\mathrm{bo}}_j$ and $v_2^{-1}\Sigma^{8j+8,1}\underline{\mathrm{bo}}_{j-1}$ in $\mathcal{D}_{A(2)_*}$. The hard part is showing that the attaching map between these two components is trivial. This is accomplished by showing that if this attaching map is non-trivial, then it is non-trivial after g-localization where g is the generator of $\pi_{20,4}^{A(2)_*}(\mathbb{F}_2)$. We then prove the g-local attaching map is trivial (see Corollary 8.5 and Theorem 9.3), strengthening the results of [BBT21].

Theorem. *There is a splitting of*

$$g^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n} \in \mathcal{D}_{A(2)_*}$$

into a well-described sum of various bigraded suspensions of

- $g^{-1}\mathbb{F}_2$,
- $g^{-1}\underline{\mathrm{bo}}_1$,
- $g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}$.

The v_2-local results of this paper may be applied to understand the TMF-resolution, where

$$\mathrm{TMF} = \mathrm{tmf}[\Delta^{-1}].$$

Namely, there are localized ASS's

$$\pi_{*,*}^{A(2)_*}(v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes s} \otimes \underline{X}) \Rightarrow \pi_*(\mathrm{TMF} \wedge \overline{\mathrm{TMF}}^{\wedge s} \wedge X)_2^{\wedge}.$$

Our v_2-local results also may be used to understand the v_2-localized algebraic tmf resolution

$$v_2^{-1}\pi_{*,*}^{A(2)_*}(\overline{\underline{\mathrm{tmf}}}^{\otimes n} \otimes M) \Rightarrow v_2^{-1}\pi_{*,*}^{A_*}(M).$$

Here, the v_2-localized Ext groups $v_2^{-1}\pi_{*,*}^{A_*}$ are as defined in [MS87].

The g-local results of this paper may be applied to understand g-local Ext over the Steenrod algebra, using the g-local algebraic tmf-resolution

$$\pi_{*,*}^{A(2)_*}(g^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n} \otimes M) \Rightarrow g^{-1}\pi_{*,*}^{A_*}(M).$$

Organization of the paper. In Section 2 we reduce the study of $\underline{\mathrm{tmf}}$ to the bo-Brown-Gitler comodules $\underline{\mathrm{bo}}_j$. We review exact sequences which relate these comodules to $\underline{\mathrm{bo}}_1^{\otimes k}$. Upon v_2-localization, we show that these exact sequences give complete decompositions of $v_2^{-1}\mathrm{bo}_j$ in terms of bigraded suspensions of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$ for various k, *provided certain obstructions $\partial_{j'}$ vanish for $j' \le j/2$.*

In Section 3 we review the structure of $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$ for $0 \le k \le 4$. These will form the computational input for the rest of the paper.

In Section 4 we construct $\underline{\mathrm{TMF}_0(3)} \in \mathcal{D}_{A(2)_*}$, our algebraic analog of $\mathrm{TMF}_0(3)$, and establish some basic properties.

In Section 5 we prove a few key splitting theorems that inductively give complete decompositions of $\underline{\mathrm{bo}}_1^{\otimes k} \in \mathcal{D}_{A(2)_*}$ into indecomposable summands. Provided the obstructions $\partial_{j'}$ vanish, we therefore get complete decompositions of $v_2^{-1}\underline{\mathrm{bo}}_j$.

In Section 6 we define certain generating functions which conveniently allow for algebraic computation of the putative decompositions of $v_2^{-1}\underline{\mathrm{bo}}_j$.

In Section 7 we explain the analogs of the v_2-local decompositions of $\underline{\mathrm{bo}}_j$ and $\underline{\mathrm{bo}}_1^{\otimes k}$ in the g-local category. The decompositions of $g^{-1}\underline{\mathrm{bo}}_j$ depend on the vanishing of certain obstructions ∂'_j.

Section 8, we prove our main result: the obstructions ∂_j and ∂'_j vanish for all j. This results in a complete decomposition of $v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}$ and $g^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}$.

In Section 9, we relate our g-local results to the computations of Bhattacharya, Bobkova, and Thomas [BBT21], providing a strengthening of their results.

In Appendix A, we discuss a stable splitting of $\mathrm{bo}_1^{\wedge 3}$ and its relationship with Theorem 1.6.

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2. bo-Brown-Gitler comodules

In this section we reduce the analysis of $v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}$ to the analysis of v_2-local bo-Brown-Gitler comodules. These are A_*-comodules which are the homology of the bo-Brown-Gitler spectra constructed by [GJM86]. Mahowald used integral Brown-Gitler spectra to analyze the bo resolution [Mah81]. The bo-Brown-Gitler comodules play a similar role in the algebraic tmf resolution [BHHM08], [MR09], [DM10], [BOSS19], [BHHM20], [BMQ21].

Endow the mod 2 homology of bo

$$\underline{\mathrm{bo}} \cong A /\!\!/ A(1)_* = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots]$$

(where ζ_i denotes the conjugate of $\xi_i \in A_*$) with a multiplicative grading by declaring the *weight* of ζ_i to be

$$wt(\zeta_i) = 2^{i-1}. \tag{2.1}$$

The ith bo-*Brown-Gitler* comodule is the subcomodule

$$\underline{\mathrm{bo}}_i \subset A/\!\!/A(1)_*$$

spanned by monomials of weight less than or equal to $4i$.

For an object $M \in \mathcal{D}_{A(2)_*}$, let

$$DM = \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$$

be its $\mathbb{F}_2$-linear dual. We record the following useful result.

Proposition 2.2. *There is an equivalence*

$$v_2^{-1} D\underline{\mathrm{bo}}_1 \simeq \Sigma^{-16,-1} v_2^{-1}\underline{\mathrm{bo}}_1.$$

Proof. This follows from the short exact sequence

$$0 \to \underline{\mathrm{bo}}_1 \to A(2)/\!\!/A(1)_* \to \Sigma^{17} D\underline{\mathrm{bo}}_1 \to 0.$$

□

Our interest in the bo-Brown-Gitler comodules stems from the fact that there is a splitting of $A(2)_*$-comodules [BHHM08, Cor. 5.5]:

$$\underline{\mathrm{tmf}} \cong \bigoplus_{i \geq 0} \Sigma^{8i} \underline{\mathrm{bo}}_i \tag{2.3}$$

where $\Sigma^{8j}\underline{\mathrm{bo}}_j$ is spanned by the monomials of

$$\underline{\mathrm{tmf}} = A/\!\!/A(2)_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]$$

of weight $8j$. We therefore have a splitting of $A(2)_*$-comodules

$$\overline{\underline{\mathrm{tmf}}}^{\otimes n} \cong \bigoplus_{i_1,\ldots,i_n > 0} \Sigma^{8(i_1+\cdots+i_n)} \underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n}. \tag{2.4}$$

The object

$$\Sigma^{8(i_1+\cdots+i_n)} \underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \in \mathcal{D}_{A(2)_*}$$

can be inductively built from $\underline{\mathrm{bo}}_1^{\otimes k}$ by means of a set of exact sequences of $A(2)_*$-comodules which relate the $\underline{\mathrm{bo}}_i$'s [BHHM08, Sec. 7]:

$$0 \to \Sigma^{8j}\underline{\mathrm{bo}}_j \to \underline{\mathrm{bo}}_{2j} \to A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \to \Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1} \to 0, \tag{2.5}$$

$$0 \to \Sigma^{8j}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \to \underline{\mathrm{bo}}_{2j+1} \to A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \to 0. \tag{2.6}$$

Here, $\underline{\mathrm{tmf}}_j$ is the jth tmf-Brown-Gitler comodule — it is the subcomodule of $\underline{\mathrm{tmf}}$ spanned by monomials of weight less than or equal to $8j$.

Remark 2.7. Technically speaking, as is addressed in [BHHM08, Sec. 7], the comodules

$$A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1}$$

in the above exact sequences have to be given a slightly different $A(2)_*$-comodule structure from the standard one arising from the tensor product. However, this different comodule structure ends up being Ext-isomorphic to the standard one. As the analysis of this paper only requires

$$v_2^{-1}A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \simeq 0,$$
$$g^{-1}A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \simeq 0,$$

and these equivalences hold for the non-standard comodule structures, the reader can safely ignore this subtlety.

Since

$$v_2^{-1}A(2)/\!\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \simeq 0,$$

The exact sequences (2.5) and (2.6) give rise to a cofiber sequence in $\mathcal{D}_{A(2)_*}$

$$\Sigma^{8j} v_2^{-1}\underline{\mathrm{bo}}_j \to v_2^{-1}\underline{\mathrm{bo}}_{2j} \to \Sigma^{8j+8,1} v_2^{-1}\underline{\mathrm{bo}}_{j-1} \tag{2.8}$$

and an equivalence

$$\Sigma^{8j} v_2^{-1}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \simeq v_2^{-1}\underline{\mathrm{bo}}_{2j+1}. \tag{2.9}$$

Thus, (2.8) and (2.9) inductively build

$$v_2^{-1}\underline{\mathrm{bo}}_i \in \mathcal{D}_{A(2)_*}$$

out of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$.

The connecting homomorphism of the cofiber sequence (2.8)

$$\partial_j : v_2^{-1}\Sigma^{8j+8,1}\underline{\mathrm{bo}}_{j-1} \to v_2^{-1}\Sigma^{8j+1,-1}\mathrm{bo}_j \tag{2.10}$$

is the obstruction to the cofiber sequence being split. We will prove in Section 8 that the connecting homomorphism $\partial_j = 0$ for all j, so we have

$$v_2^{-1}\underline{\mathrm{bo}}_{2j} \simeq v_2^{-1}\Sigma^{8j}\underline{\mathrm{bo}}_j \oplus v_2^{-1}\Sigma^{8j+8,1}\underline{\mathrm{bo}}_{j-1}. \tag{2.11}$$

3. The groups $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^k)$

In the previous section we related the comodules $\underline{\mathrm{bo}}_j$ to the comodules $\underline{\mathrm{bo}}_1^{\otimes k}$. We now review the structure of

$$\pi_{*,*}^{A(2)_*}\underline{\mathrm{bo}}_1^{\otimes k}$$

for $0 \le k \le 4$.

In order to give names to the v_0-torsion-free generators of $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$, we review the corresponding v_0-local computations. The entire structure of the v_0-local algebraic tmf resolution is given in [BMQ21] (see also [BOSS19]).

Observe that we have

$$v_0^{-1}\pi_{*,*}^{A(2)_*}(\mathbb{F}_2) = \mathbb{F}_2[v_0^{\pm}, v_1^4, v_2^2]. \tag{3.1}$$

Note that $c_4, c_6 \in (\mathrm{tmf}_*)_{\mathbb{Q}}$ are detected in the v_0-localized ASS by v_1^4 and $v_0^3 v_2^2$, respectively.

We have (regarding $\underline{\mathrm{bo}}_1$ as a subcomodule of $A/\!\!/A(2)_*$)

$$v_0^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1) = \mathbb{F}_2[v_0^{\pm}, v_1^4, v_2^2]\{\bar{\xi}_1^8, \bar{\xi}_2^4\}$$

We therefore have an isomorphism

$$v_0^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k}) \cong \mathbb{F}_2[v_0^{\pm}, v_1^4, v_2^2] \otimes \mathbb{F}_2\{\bar{\xi}_1^8, \bar{\xi}_2^4\}^{\otimes k}. \tag{3.2}$$

To make for more compact notation, we will use bars to denote elements of tensor powers:

$$x_1|\cdots|x_n := x_1 \otimes \cdots \otimes x_n. \tag{3.3}$$

$\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$: (Figure 3.1)

All of the elements are $c_4 = v_1^4$-periodic, and v_2^8-periodic. Exactly one v_1^4 multiple of each element is displayed with the $\bullet$ replaced by a $\circ$. Observe the wedge pattern beginning in $t - s = 35$. This pattern is infinite, propagated horizontally by $h_{2,1}$-multiplication and vertically by v_1-multiplication. Here, $h_{2,1}$ is the name of the generator in the May spectral sequence of bidegree $(t-s, s) = (5, 1)$, and $h_{2,1}^4 = g$.

$\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$, for $k = 1, 2, 3, 4$: (Figures 3.2, 3.3, 3.4, 3.5)

Every element is v_2^8-periodic. However, unlike $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$, not every element of these Ext groups is v_1^4-periodic. Rather, it is the case that either an element $x \in \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$ satisfies $v_1^4 x = 0$, or it is v_1^4-periodic. Each of the v_1^4-periodic elements fit into families which look like shifted and truncated copies of $\pi_{*,*}^{A(1)_*}(\mathbb{F}_2)$, and are labeled with a $\circ$. We have only included the beginning of these v_1^4-periodic patterns in the chart. The other generators are labeled with a $\bullet$. A $\square$ indicates a polynomial algebra $\mathbb{F}_2[h_{2,1}]$. Elements which are v_0-torsion-free are named in these charts using (3.2), in the bar notation of (3.3).

4. An algebraic model of $\mathrm{TMF}_0(3)$

The spectrum $\mathrm{TMF}_0(3)$ is an analog of TMF associated to the moduli of elliptic curves with with $\Gamma_0(3)$-structures introduced and studied by Mahowald and Rezk [MR09]. In fact, Mahowald and Rezk proposed three different connective spectra whose $E(2)$-localizations are $\mathrm{TMF}_0(3)$ (also see [DM10]).

We will emulate [MR09, DM10] in the category of $\mathcal{D}_{A(2)_*}$ to construct the $\underline{\mathrm{TMF}_0(3)}$.

Lemma 4.1. *The composite*

$$\Sigma^{6,2}\mathbb{F}_2 \xrightarrow{h_2^2} \mathbb{F}_2 \hookrightarrow \Sigma^7 D\underline{\mathrm{bo}}_1$$

extends to a map

$$\widetilde{h_2^2} : \Sigma^{6,2}\underline{\mathrm{bo}}_1 \to \Sigma^7 D\underline{\mathrm{bo}}_1.$$

Our algebraic model of $\mathrm{TMF}_0(3)$ is defined to be

$$\underline{\mathrm{TMF}_0(3)} := v_2^{-1}(\Sigma^{24,3}D\underline{\mathrm{bo}}_1 \cup_{\widetilde{h_2^2}} \Sigma^{24,4}\underline{\mathrm{bo}}_1).$$

Figure 4.1 shows a computation of the homotopy of $D\underline{\mathrm{bo}}_1 \cup_{\widetilde{h_2^2}} \Sigma^{0,1}\underline{\mathrm{bo}}_1$. In this figure, the solid dots correspond to $D\underline{\mathrm{bo}}_1$ and the open dots correspond to $\underline{\mathrm{bo}}_1$. One convenient way of accessing the homotopy of $D\underline{\mathrm{bo}}_1$ is from the short exact

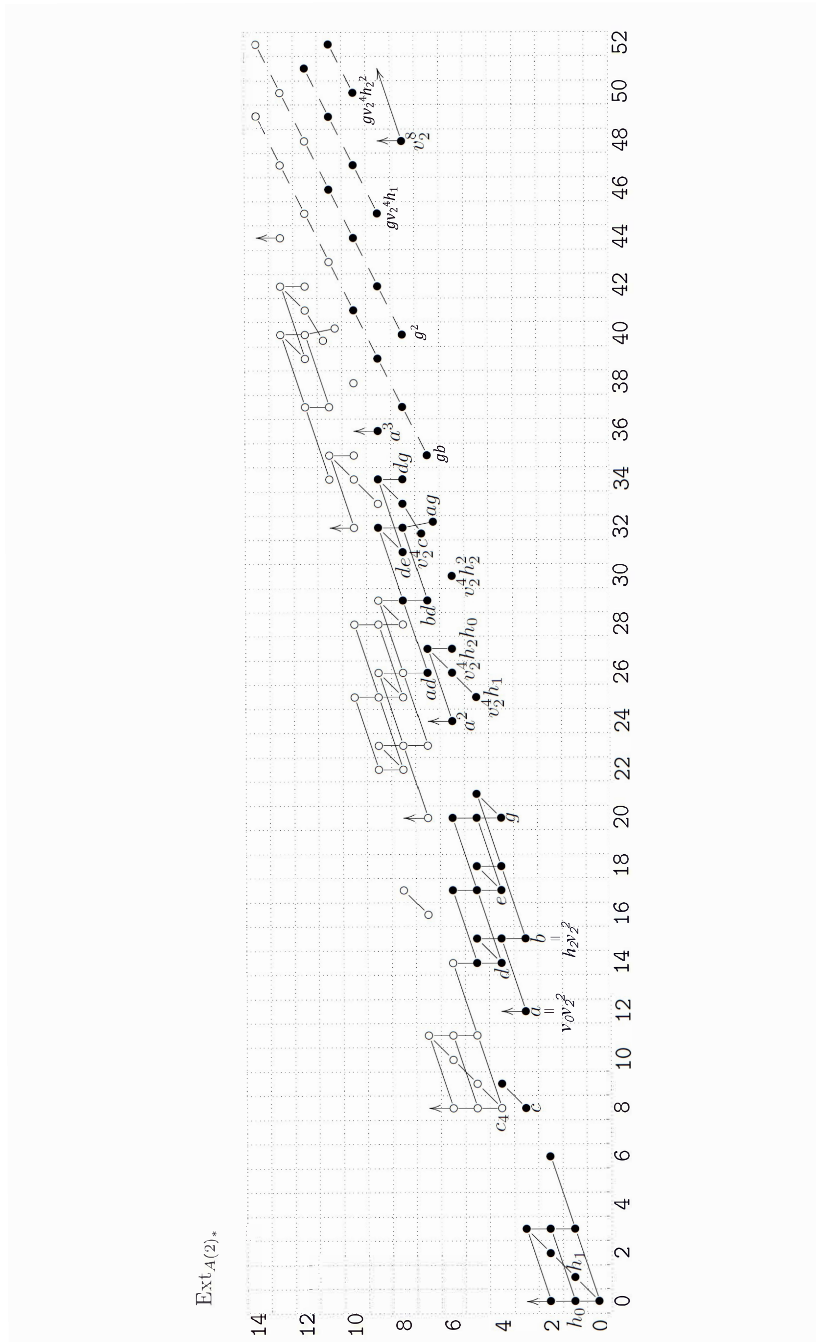

FIGURE 3.1. $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$.

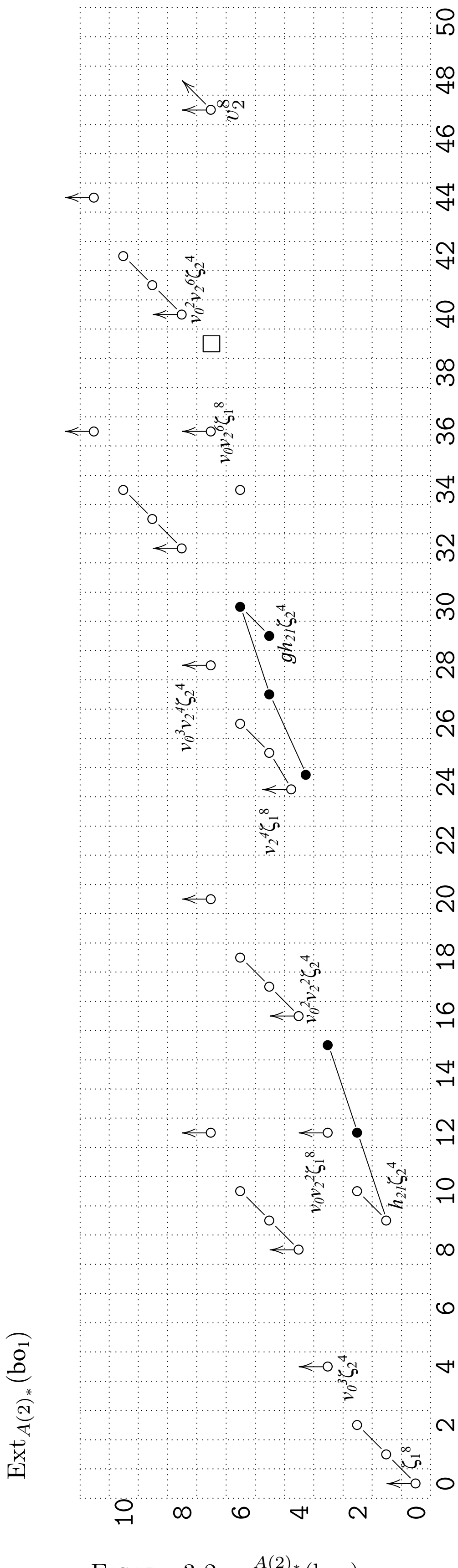

FIGURE 3.2. $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1)$.

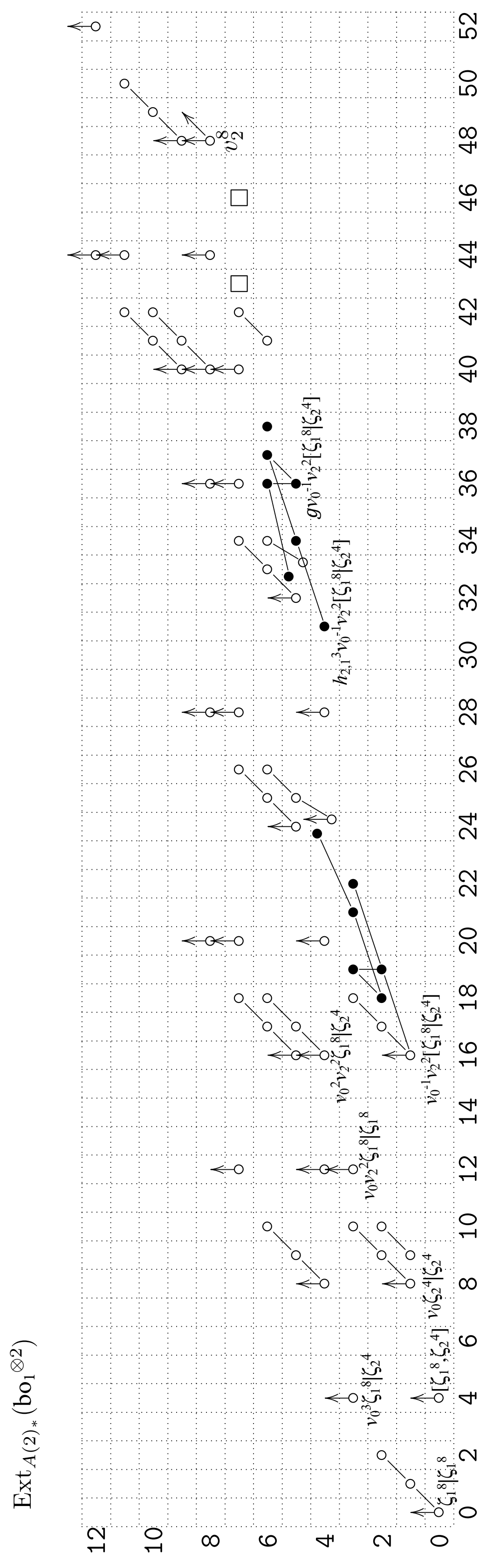

FIGURE 3.3. $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2})$.

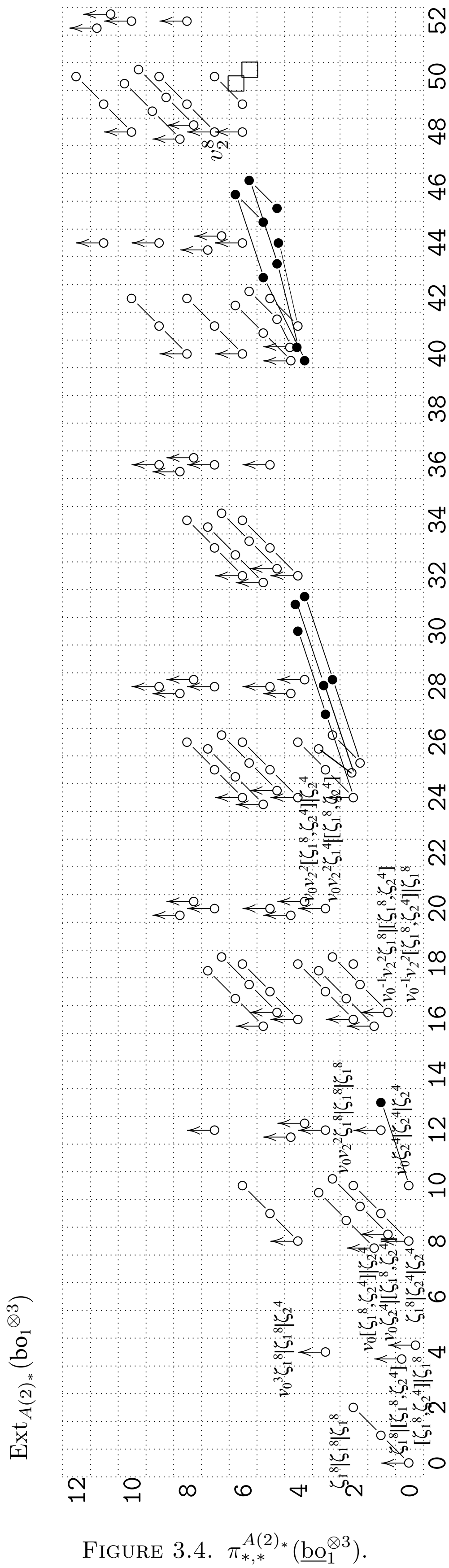

FIGURE 3.4. $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3})$.

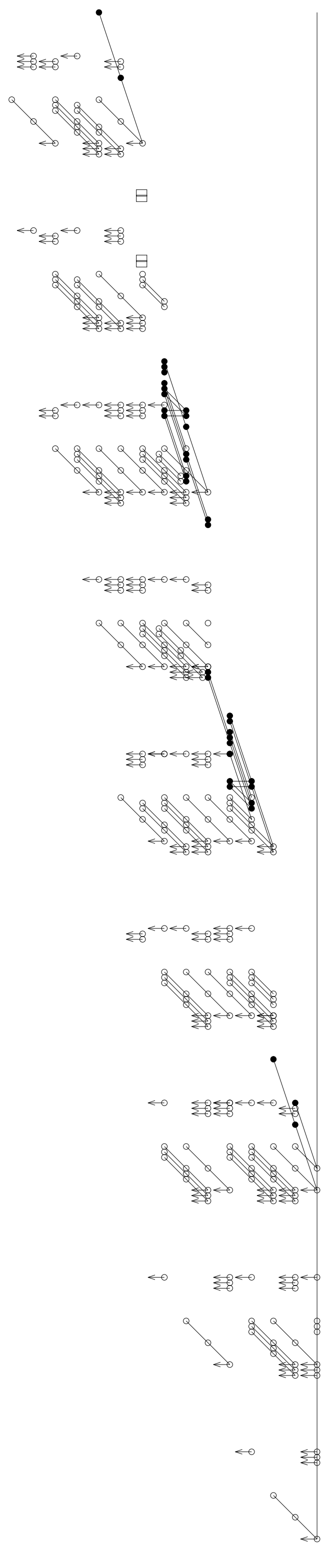

FIGURE 3.5. $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 4})$.

sequence in the proof of Proposition 2.2. A chart of $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)})$ is displayed in Figure 4.2.

Lemma 4.2. *Any map*

$$f : \underline{\mathrm{TMF}_0(3)} \to \underline{\mathrm{TMF}_0(3)}$$

which is the identity on $\pi_{0,0}^{A(2)_*}$ *is an equivalence.*

Proof. Let $1_{\underline{\mathrm{TMF}_0(3)}} \in \pi_{0,0}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)})$ denote the generator. The $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$-module structure implies f is the identity on $g \cdot 1_{\underline{\mathrm{TMF}_0(3)}}$ and $v_2^4 h_1$. It follows from h_2 linearity that f is the identity on x_{17} (see Figure 4.2). Therefore f is the identity on $v_2^4 h_1 x_{17}$. It follows from h_0, h_1, h_2, and v_1^4 linearity that f is an isomorphism on $v_0^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)})$. Here we must use the fact that the v_0-localization of f is a map of $v_0^{-1}\pi_{*,*}(\mathbb{F}_2)$-modules. It then follows that f is a $\pi_{*,*}^{A(2)_*}$-isomorphism. □

We have the following algebraic version of the Recognition Principle of Davis-Mahowald-Rezk (see [MR09, Prop. 7.2]).

Theorem 4.3 (Recognition Principle)**.** *Suppose that* $X \in \mathcal{D}_{A(2)_*}$ *satisfies*

$$\pi_{*,*}^{A(2)_*}(X) \cong \pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)}) \tag{4.4}$$

where the above isomorphism preserves v_0, h_1, h_2, v_1^4, $v_0 v_2^2$, v_2^8, $v_2^4 h_1$, *and* g *multiplications. Then there is an equivalence*

$$X \simeq \underline{\mathrm{TMF}_0(3)}.$$

Proof. Let

$$x_{17} : \Sigma^{17,3}\mathbb{F}_2 \to X$$

represent the generator of $\pi_{17,3}^{A(2)_*}(X)$. Since

$$\pi_{17,4}^{A(2)_*}(X) = \pi_{19,4}^{A(2)_*}(X) = \pi_{23,4}^{A(2)_*}(X) = 0,$$

there exists an extension of x_{17} to a map

$$\Sigma^{24,3} D\mathrm{bo}_1 \to X.$$

Since

$$\pi_{23,5}^{A(2)_*}(X) = \pi_{27,5}^{A(2)_*}(X) = \pi_{29,5}^{A(2)_*}(X) = \pi_{30,5}^{A(2)_*}(X) = 0$$

there exists a further extension of this map to a map

$$\Sigma^{24,3} D\underline{\mathrm{bo}}_1 \cup \Sigma^{24,4}\underline{\mathrm{bo}}_1 \to X.$$

The conditions on the isomorphism (4.4) imply that $X \simeq v_2^{-1}X$. Thus the map above localizes to a map

$$v_2^{-1}(\Sigma^{24,3} D\underline{\mathrm{bo}}_1 \cup \Sigma^{24,4}\underline{\mathrm{bo}}_1) \to X.$$

The conditions on the isomorphism (4.4) then force the map above to be a $\pi_{*,*}^{A(2)_*}$-isomorphism. □

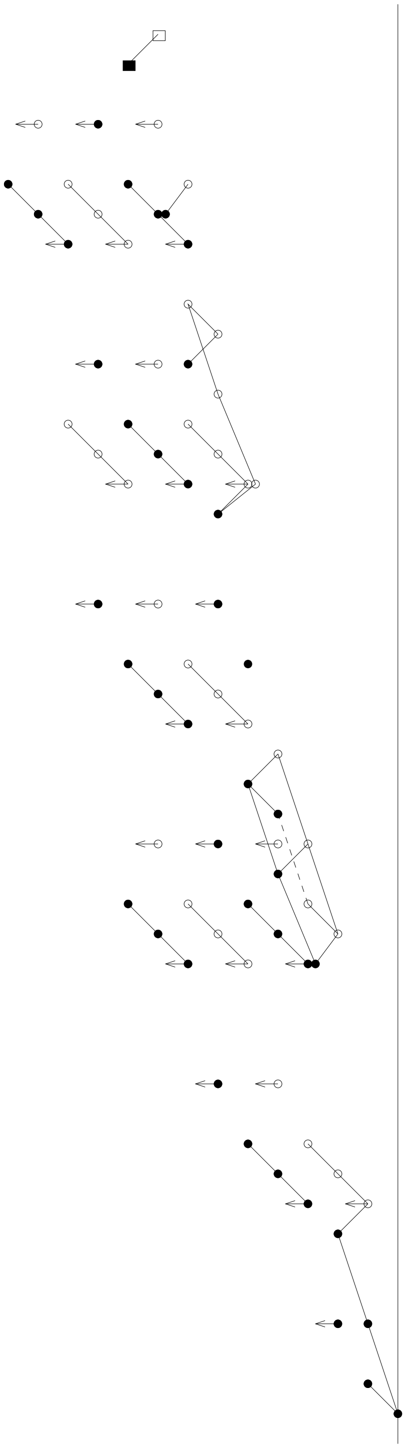

FIGURE 4.1. Computing the homotopy of $D\underline{\mathrm{bo}}_1 \cup_{\widetilde{h_2^2}} \Sigma^{0,1}\underline{\mathrm{bo}}_1$.

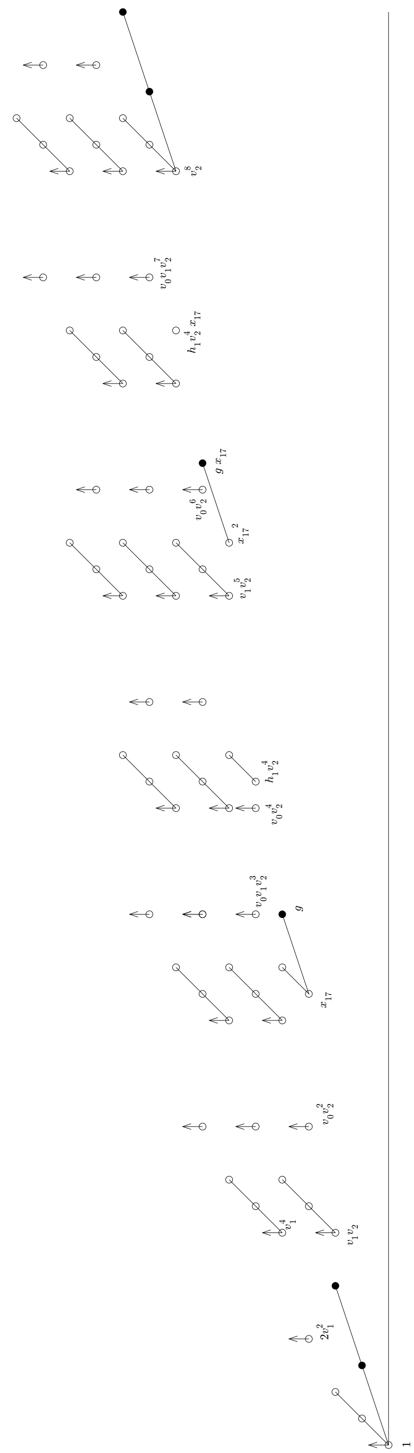

FIGURE 4.2. $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)})$.

For us, *a weak ring object* in $\mathcal{D}_{A(2)_*}$ is an object $R \in \mathcal{D}_{A(2)_*}$ with a unit
$$u : \mathbb{F}_2 \to R$$
and a multiplication
$$m : R \otimes R \to R$$
such that the two composites
$$R \otimes \mathbb{F}_2 \xrightarrow{1 \otimes u} R \otimes R \xrightarrow{m} R,$$
$$\mathbb{F}_2 \otimes R \xrightarrow{u \otimes 1} R \otimes R \xrightarrow{m} R$$
are equivalences.

Proposition 4.5. $\underline{\mathrm{TMF}_0(3)}$ *is a weak ring object in* $\mathcal{D}_{A(2)_*}$.

Proof. We shall need to imitate the "first model" of [MR09], [DM10]. Start with the A_*-comodule $\underline{Y}$ described in [DM10, Thm. 2.1(a)]. Then the method of proof for [DM10, Thm. 2.1(b)] shows that there exists a map
$$\widetilde{h_0 h_2} : \Sigma^{3,2}\underline{Y} \to \mathbb{F}_2$$
in $\mathcal{D}_{A_*}$ extending h_0h_2, so we can take the cofiber
$$\underline{X} := \mathbb{F}_2 \cup_{\widetilde{h_0 h_2}} \Sigma^{4,1}\underline{Y}.$$
Regarding this cofiber as an object of $\mathcal{D}_{A(2)_*}$, define
$$R := v_2^{-1}\underline{X} \in \mathcal{D}_{A(2)_*}.$$
We will show (a) $R \simeq \underline{\mathrm{TMF}_0(3)}$ and (b) R is a ring object of $\mathcal{D}_{A(2)_*}$.

For (a), we will compute $\pi^{A(2)_*}_{*,*}(R)$. To this end, we observe that the methods of the proof of [DM10, Thm. 2.1(c)] show that there is a map
$$f : \underline{X} \to A(2) /\!\!/ A(1)_*$$
which extends the inclusion $\mathbb{F}_2 \hookrightarrow A(2)/\!\!/A(1)_*$. Let $\underline{C}$ be the cofiber of f:
$$\underline{X} \xrightarrow{f} A(2)/\!\!/A(1)_* \to \underline{C}. \tag{4.6}$$
Then the proof of [DM10, Thm. 2.1(d)] shows that
$$\pi^{A(2)_*}_{*,s}(A(2)_* \otimes \underline{C}) \cong \begin{cases} \Sigma^4 A(2)/A(2)(\mathrm{Sq}^4, \mathrm{Sq}^5\,\mathrm{Sq}^1)_*, & s = 0, \\ 0, & s > 0. \end{cases}$$
as an $A(2)_*$-comodule. The $A(2)_*$-based Adams spectral sequence for $\underline{C}$ then collapses to give an isomorphism
$$\pi^{A(2)_*}_{n,s}(\underline{C}) \cong \mathrm{Ext}^{s+n,s}_{A(2)_*}(\mathbb{F}_2, \Sigma^4 A(2)/A(2)(\mathrm{Sq}^4, \mathrm{Sq}^5\,\mathrm{Sq}^1)_*).$$
These Ext groups were computed in [DM10, Thm. 2.9]. The cofiber sequence (4.6) gives an equivalence
$$R \simeq \Sigma^{-1,1}v_2^{-1}\underline{C}.$$
We see by inspection of Davis-Mahowald's Ext computation alluded to above that there is an isomorphism
$$\pi^{A(2)}_{*,*}(\Sigma^{-1,1}v_2^{-1}\underline{C}) \cong \pi^{A(2)_*}_{*,*}(\underline{\mathrm{TMF}_0(3)})$$

satisfying the hypotheses of the Recognition Principle (Theorem 4.3). We deduce that there is an equivalence

$$\underline{\mathrm{TMF}_0(3)} \simeq R.$$

We now just need to prove R is a ring object in $\mathcal{D}_{A(2)_*}$. For this we imitate the proof of [DM10, Thm. 2.1(e)]. Namely, consider the composite

$$\overline{m} : \underline{X} \otimes \underline{X} \xrightarrow{f \otimes f} A(2) /\!\!/ A(1)_* \otimes A(2) /\!\!/ A(1)_* \xrightarrow{\mu} A(2) /\!\!/ A(1)_*.$$

By the cofiber sequence (4.6), the map $\overline{m}$ lifts to a map

$$m : \underline{X} \otimes \underline{X} \to \underline{X}$$

if the composite

$$\underline{X} \otimes \underline{X} \xrightarrow{\overline{m}} A(2) /\!\!/ A(1)_* \to \underline{C}$$

is null. In the proof of [DM10, Thm. 2.1(e)], it is established using Bruner's Ext software that

$$[\underline{X} \otimes \underline{X}, \underline{C}]_{A(2)_*} = 0.$$

Therefore, the lift m exists. Since it is a lift of $\overline{m}$, it is the identity on the bottom cell. It follows that the composites

$$\underline{X} \otimes \mathbb{F}_2 \hookrightarrow \underline{X} \otimes \underline{X} \xrightarrow{m} \underline{X},$$
$$\mathbb{F}_2 \otimes \underline{X} \hookrightarrow \underline{X} \otimes \underline{X} \xrightarrow{m} \underline{X}$$

are the identity on the bottom cell. It follows from Lemma 4.2 that after v_2-localization, the composites

$$R \otimes \mathbb{F}_2 \hookrightarrow R \otimes R \xrightarrow{m} R,$$
$$\mathbb{F}_2 \otimes R \hookrightarrow R \otimes R \xrightarrow{m} R$$

are equivalences. Thus m gives R the structure of a weak ring object. (In fact, the analog of Lemma 4.2 holds for $\underline{X}$, and so $\underline{X}$ is also a weak ring object.) □

5. Splitting $\underline{\mathrm{bo}}_1^{\otimes k}$

In this section we prove our main v_2-local splitting theorems, which will be the basis of all of our subsequent v_2-local decomposition results.

Proposition 5.1. *There is a splitting*

$$v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 3} \simeq 2\Sigma^{16,1} v_2^{-1}\underline{\mathrm{bo}}_1 \oplus \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)}.$$

Proof. Since we are working in characteristic 2, there is a decomposition

$$\underline{\mathrm{bo}}_1^{\otimes 3} \simeq (\underline{\mathrm{bo}}_1^{\otimes 3})^{hC_3} \oplus B$$

where C_3 acts by cyclically permuting the terms, and we have

$$\pi_{*,*}^{A(2)_*}((\underline{\mathrm{bo}}_1^{\otimes 3})^{hC_3}) = \pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3})^{C_3}.$$

It is easily checked, using the names of the generators in Figure 3.4, that there is an isomorphism

$$v_2^{-1}\pi_{*,*}^{A(2)_*}((\underline{\mathrm{bo}}_1^{\otimes 3})^{hC_3}) \cong \pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)}).$$

A direct application of the Recognition Principle (Theorem 4.3) shows that

$$v_2^{-1}(\underline{\mathrm{bo}}_1^{\otimes 3})^{hC_3} \simeq \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)}.$$

Let

$$x_{16} : \Sigma^{16,1}\mathbb{F}_2 \to \underline{\mathrm{bo}}_1^{\otimes 2}$$

correspond to the generator of $\pi^{A(2)_*}_{16,1}(\underline{\mathrm{bo}}_1^{\otimes 2})$. Then the composite

$$\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1 \oplus \Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1 \xrightarrow{x_{16}\otimes 1 \oplus 1\otimes x_{16}} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 3} \to v_2^{-1}B$$

is seen to be a $\pi^{A(2)_*}_{*,*}$-isomorphism, hence an equivalence. $\square$

Proposition 5.2. *There is a splitting*

$$\underline{\mathrm{TMF}_0(3)} \wedge \underline{\mathrm{bo}}_1 \simeq \Sigma^{24,3}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{40,6}\underline{\mathrm{TMF}_0(3)}.$$

Proof. Tensoring the splitting of Proposition 5.1 with $\underline{\mathrm{bo}}_1$, we have

$$v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 4} \simeq 2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)} \wedge \underline{\mathrm{bo}}_1.$$

Examination of $\pi^{A(2)_*}_{*,*}(\underline{\mathrm{bo}}_1^{\otimes 4})$ (Figure 3.5) reveals that

$$\pi^{A(2)_*}_{*,*}(v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 4}) \simeq \\ 2\pi^{A(2)_*}_{*,*}(\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \oplus \pi^{A(2)_*}_{*,*}(\Sigma^{48,5}\underline{\mathrm{TMF}_0(3)}) \oplus \pi^{A(2)_*}_{*,*}(\Sigma^{64,8}\underline{\mathrm{TMF}_0(3)}).$$

It follows that there is an isomorphism

$$\pi^{A(2)_*}_{*,*}(\underline{\mathrm{TMF}_0(3)} \wedge \underline{\mathrm{bo}}_1) \cong \pi^{A(2)_*}_{*,*}(\Sigma^{24,3}\underline{\mathrm{TMF}_0(3)}) \oplus \pi^{A(2)_*}_{*,*}(\Sigma^{40,6}\underline{\mathrm{TMF}_0(3)}).$$

Moreover, one can check form the $\pi^{A(2)_*}_{*,*}(\mathbb{F}_2)$-module structure of $\pi^{A(2)_*}_{*,*}(\underline{\mathrm{bo}}_1^{\otimes 4})$ that the isomorphism preserves multiplication by

$$v_0, v_1^4, v_0v_2^2, v_2^8, h_1, h_2, g, v_2^4h_1.$$

The map

$$\Sigma^{24,3}\mathbb{F}_2 \oplus \Sigma^{40,6}\mathbb{F}_2 \to \underline{\mathrm{TMF}_0(3)} \wedge \underline{\mathrm{bo}}_1$$

which maps the two generators in gives rise to a map of $\underline{\mathrm{TMF}_0(3)}$-modules

$$\Sigma^{24,3}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{40,6}\underline{\mathrm{TMF}_0(3)} \to \underline{\mathrm{TMF}_0(3)} \wedge \underline{\mathrm{bo}}_1.$$

One can then use $\pi^{A(2)_*}_{*,*}(\mathbb{F}_2)$-module structures to determine that this map is an isomorphism on $\pi^{A(2)_*}_{*,*}$. $\square$

Remark 5.3. Propositions 5.1 and 5.2 allow one to inductively compute a splitting of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$ in $\mathcal{D}_{A(2)_*}$ as a sum of suspensions of $v_2^{-1}\underline{\mathrm{bo}}_1$, $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}$ and $\underline{\mathrm{TMF}_0(3)}$. For example, we have

$$\begin{aligned} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 4} &\simeq (2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1 \oplus \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)}) \otimes \underline{\mathrm{bo}}_1 \\ & 2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus \Sigma^{24,2}\underline{\mathrm{TMF}_0(3)} \otimes \underline{\mathrm{bo}}_1 \\ & 2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus \Sigma^{48,5}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{64,8}\underline{\mathrm{TMF}_0(3)}. \end{aligned}$$

In the next case, we can further simplify the answer using v_2^8 periodicity.

$$\begin{aligned}
v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 5} &\simeq (2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus \Sigma^{48,5}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{64,8}\underline{\mathrm{TMF}_0(3)}) \otimes \underline{\mathrm{bo}}_1 \\
&\simeq 2\Sigma^{16,1}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 3} \oplus \Sigma^{48,5}\underline{\mathrm{TMF}_0(3)} \otimes \underline{\mathrm{bo}}_1 \oplus \Sigma^{64,8}\underline{\mathrm{TMF}_0(3)} \otimes \underline{\mathrm{bo}}_1 \\
&\simeq 4\Sigma^{32,2}v_2^{-1}\underline{\mathrm{bo}}_1 \oplus 2\Sigma^{40,3}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{72,8}\underline{\mathrm{TMF}_0(3)} \\
&\quad \oplus 2\Sigma^{88,11}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{104,14}\underline{\mathrm{TMF}_0(3)} \\
&\simeq 4\Sigma^{32,2}v_2^{-1}\underline{\mathrm{bo}}_1 \oplus \Sigma^{24}\underline{\mathrm{TMF}_0(3)} \oplus 4\Sigma^{40,3}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{56,6}\underline{\mathrm{TMF}_0(3)}.
\end{aligned}$$

We similarly may compute

$$\begin{aligned}
v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 6} \simeq 4\Sigma^{32,2}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} &\oplus \Sigma^{48,3}\underline{\mathrm{TMF}_0(3)} \oplus 5\Sigma^{64,6}\underline{\mathrm{TMF}_0(3)} \\
&\oplus 5\Sigma^{32,1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{48,4}\underline{\mathrm{TMF}_0(3)}.
\end{aligned} \tag{5.4}$$

Finally, we will find the following splitting to be useful.

Proposition 5.5. *There is a splitting*

$$\underline{\mathrm{TMF}_0(3)}^{\otimes 2} \simeq \underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{0,-1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{16,2}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{32,5}\underline{\mathrm{TMF}_0(3)}.$$

Proof. Smashing the splitting of Proposition 5.1 with itself, and applying Proposition 5.2 and v_2^8-periodicity, we have

$$\begin{aligned}
v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 6} &\simeq 4\Sigma^{32,2}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus 4\Sigma^{40,3}\underline{\mathrm{bo}}_1 \otimes \underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{48,4}\underline{\mathrm{TMF}_0(3)}^{\otimes 2} \\
&\simeq 4\Sigma^{32,2}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus 4\Sigma^{64,6}\underline{\mathrm{TMF}_0(3)} \oplus 4\Sigma^{80,9}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{48,4}\underline{\mathrm{TMF}_0(3)}^{\otimes 2} \\
&\simeq 4\Sigma^{32,2}\underline{\mathrm{bo}}_1^{\otimes 2} \oplus 4\Sigma^{64,6}\underline{\mathrm{TMF}_0(3)} \oplus 4\Sigma^{32,1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{48,4}\underline{\mathrm{TMF}_0(3)}^{\otimes 2}.
\end{aligned}$$

On the other hand, by (5.4), we have

$$\begin{aligned}
v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 6} \simeq 4\Sigma^{32,2}v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} &\oplus \Sigma^{48,3}\underline{\mathrm{TMF}_0(3)} \oplus 5\Sigma^{64,6}\underline{\mathrm{TMF}_0(3)} \\
&\oplus 5\Sigma^{32,1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{48,4}\underline{\mathrm{TMF}_0(3)}.
\end{aligned}$$

Making use of $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$ module structures, we deduce that there is an isomorphism

$$\begin{aligned}
&\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)}^{\otimes 2}) \cong \\
&\quad \pi_{*,*}^{A(2)_*}(\Sigma^{0,-1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{16,2}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{-16,-3}\underline{\mathrm{TMF}_0(3)} \oplus \underline{\mathrm{TMF}_0(3)}) \\
&\quad\quad \cong \pi_{*,*}^{A(2)_*}(\Sigma^{0,-1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{16,2}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{32,5}\underline{\mathrm{TMF}_0(3)} \oplus \underline{\mathrm{TMF}_0(3)})
\end{aligned}$$

of $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$-modules. Since $\underline{\mathrm{TMF}_0(3)}^{\otimes 2}$ is a $\underline{\mathrm{TMF}_0(3)}$-module, we can extend the $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)})$-module generators of $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)}^{\otimes 2})$ to a map

$$\Sigma^{0,-1}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{16,2}\underline{\mathrm{TMF}_0(3)} \oplus \Sigma^{32,5}\underline{\mathrm{TMF}_0(3)} \oplus \underline{\mathrm{TMF}_0(3)} \to \underline{\mathrm{TMF}_0(3)}^{\otimes 2}$$

which is a $\pi_{*,*}^{A(2)_*}$-isomorphism, hence an equivalence. □

6. Generating functions

In this section we will describe a useful combinatorial way of computing decompositions of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$ and $v_2^{-1}\underline{\mathrm{bo}}_j$.

We will represent the objects of $\mathcal{D}_{A(2)_*}$ of the form

$$\Sigma^{8i_1,j_1} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k_1} \otimes \underline{\mathrm{TMF}_0(3)}^{\otimes l_1} \oplus \cdots \oplus \Sigma^{8i_n,j_n} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k_n} \otimes \underline{\mathrm{TMF}_0(3)}^{\otimes l_n} \tag{6.1}$$

by elements of $\mathbb{Z}[s^\pm, t^\pm, x, y]$:

$$t^{i_1}s^{j_1}x^{k_1}y^{l_1} + \cdots + t^{i_n}s^{j_n}x^{k_n}y^{l_n}.$$

Propositions 5.1, 5.2, and v_2-periodicity impose some relations on this polynomial ring — we therefore work in the quotient ring

$$R := \mathbb{Z}[s^\pm, t^\pm, x, y]/(x^3 = 2t^2sx + t^3s^2y,\ xy := t^3s^3y + t^5s^6y,\ t^6s^8 = 1). \tag{6.2}$$

Note that these relations imply

$$y^2 = y + s^{-1}y + t^2s^2y + t^4s^5y.$$

This relation reflects the splitting of Prop 7.3.

We may use the relations of R to reduce x^k to a sum of monomials whose terms are of the form t^is^jx, $t^is^jx^2$, and t^is^jy. These reduced forms of x^k correspond to splittings of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$. For example, the splitting (5.4) corresponds to the expression

$$x^6 = 5s^6t^8y + s^4t^6y + s^3t^6y + 5st^4y + 4s^2t^4x^2$$

in R. Table 1 shows the reduced forms of x^k in R for $k \le 16$.

In light of Propositions 2.2 we can also compute the duals of objects of the form (6.1) represented as an element of R via the ring map:

$$\begin{aligned} D : R &\to R \\ t &\mapsto t^{-1} \\ s &\mapsto s^{-1} \\ x &\mapsto t^{-2}s \cdot x \\ y &\mapsto s \cdot y \end{aligned}$$

Note the formula $D(y) = sy$ is forced by the relations of R. We note however that Proposition 5.1 and Proposition 2.2 can be used to deduce that $v_2^{-1}D\underline{\mathrm{TMF}_0(3)} \simeq \Sigma^{0,1}\underline{\mathrm{TMF}_0(3)}$.

Now assume that the connecting morphisms ∂_j (2.10) are trivial for for $1 \le j \le j_0$. (We will eventually prove ∂_j is always zero in Theorem 8.1.) Then we can inductively define elements of R which encode the splitting of $v_2^{-1}\underline{\mathrm{bo}}_j$ for $j \le 2j_0+1$. These are the *bo-Brown-Gitler* polynomials, introduced in [BHHM20, Sec. 8]. Their

$$\begin{array}{rcl}
x^3 & = & s^2t^3y + 2st^2x \\
x^4 & = & s^5t^6y + t^2y + 2st^2x^2 \\
x^5 & = & s^6t^7y + 4s^3t^5y + t^3y + 4s^2t^4x \\
x^6 & = & 5s^6t^8y + s^4t^6y + s^3t^6y + 5st^4y + 4s^2t^4x^2 \\
x^7 & = & 6s^7t^9y + s^6t^9y + 14s^4t^7y + s^2t^5y + 6st^5y + 8s^3t^6x \\
x^8 & = & 20s^7t^{10}y + 7s^5t^8y + 7s^4t^8y + 20s^2t^6y + st^6y + t^4y + 8s^3t^6x^2 \\
x^9 & = & 8s^7t^{11}y + s^6t^9y + 48s^5t^9y + s^4t^9y + 8s^3t^7y + 27s^2t^7y + 27t^5y \\
 & & +16s^4t^8x \\
x^{10} & = & s^7t^{12}y + 35s^6t^{10}y + 35s^5t^{10}y + s^4t^8y + 75s^3t^8y + 9s^2t^8y \\
 & & +9st^6y + 75t^6y + 16s^4t^8x^2 \\
x^{11} & = & 10s^7t^{11}y + 166s^6t^{11}y + 10s^5t^{11}y + 44s^4t^9y + 110s^3t^9y + s^2t^9y \\
 & & +s^2t^7y + 110st^7y + 44t^7y + 32s^5t^{10}x \\
x^{12} & = & 154s^7t^{12}y + 154s^6t^{12}y + s^5t^{12}y + 11s^5t^{10}y + 276s^4t^{10}y \\
 & & +54s^3t^{10}y + 54s^2t^8y + 276st^8y + 11t^8y + t^6y + 32s^5t^{10}x^2 \\
x^{13} & = & 584s^7t^{13}y + 65s^6t^{13}y + s^6t^{11}y + 208s^5t^{11}y + 430s^4t^{11}y \\
 & & +12s^3t^{11}y + 12s^3t^9y + 430s^2t^9y + 208st^9y + t^9y + 65t^7y + 64s^6t^{12}x \\
x^{14} & = & 638s^7t^{14}y + 13s^6t^{14}y + 77s^6t^{12}y + 1014s^5t^{12}y + 273s^4t^{12}y \\
 & & +s^3t^{12}y + s^4t^{10}y + 273s^3t^{10}y + 1014s^2t^{10}y + 77st^{10}y + 13st^8y + 638t^8y \\
 & & +64s^6t^{12}x^2 \\
x^{15} & = & 350s^7t^{15}y + s^6t^{15}y + 14s^7t^{13}y + 911s^6t^{13}y + 1652s^5t^{13}y \\
 & & +90s^4t^{13}y + 90s^4t^{11}y + 1652s^3t^{11}y + 911s^2t^{11}y + 14st^{11}y + s^2t^9y \\
 & & +350st^9y + 2092t^9y + 128s^7t^{14}x \\
x^{16} & = & 104s^7t^{16}y + 440s^7t^{14}y + 3744s^6t^{14}y + 1261s^5t^{14}y + 15s^4t^{14}y \\
 & & +15s^5t^{12}y + 1261s^4t^{12}y + 3744s^3t^{12}y + 440s^2t^{12}y + st^{12}y + 104s^2t^{10}y \\
 & & +2563st^{10}y + 2563t^{10}y + t^8y + 128s^7t^{14}x^2
\end{array}$$

TABLE 1. Reduced expressions for x^k in R corresponding to decompositions of $v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes k}$.

definition comes from (2.9) and (2.11).

$$\begin{aligned}
f_0 &:= 1, \\
f_1 &:= x, \\
f_{2j+1} &:= t^j x \cdot f_j, \\
f_{2j} &:= t^j f_j + t^{j+1} s \cdot f_{j-1}.
\end{aligned} \tag{6.3}$$

Table 2 shows reduced expressions for f_j in R for $j \leq 16$.

7. g-LOCAL COMPUTATIONS

We will now consider the g-local bo-Brown-Gitler comodules, for

$$g = h_{2,1}^4 \in \pi_{20,4}^{A(2)_*}(\mathbb{F}_2).$$

The g-local results of this section will be crucial for the main result of Section 8.

$$
\begin{aligned}
f_1 &= x \\
f_2 &= tx + st^2 \\
f_3 &= tx^2 \\
f_4 &= st^3x + t^3x + st^4 \\
f_5 &= t^3x^2 + st^4x \\
f_6 &= t^4x^2 + st^5x + s^2t^6 \\
f_7 &= s^2t^7y + 2st^6x \\
f_8 &= st^6x^2 + st^7x + t^7x + st^8 \\
f_9 &= st^7x^2 + t^7x^2 + st^8x \\
f_{10} &= t^8x^2 + s^2t^9x + 2st^9x + s^2t^{10} \\
f_{11} &= s^2t^{11}y + st^9x^2 + 2st^{10}x \\
f_{12} &= st^{10}x^2 + t^{10}x^2 + s^2t^{11}x + st^{11}x + s^2t^{12} \\
f_{13} &= s^2t^{13}y + st^{11}x^2 + s^2t^{12}x + 2st^{12}x \\
f_{14} &= s^2t^{14}y + st^{12}x^2 + s^2t^{13}x + 2st^{13}x + s^3t^{14} \\
f_{15} &= s^5t^17y + t^{13}y + 2st^{13}x^2 \\
f_{16} &= s^3t^{16}y + st^{14}x^2 + 2s^2t^{15}x + st^{15}x + t^{15}x + st^{16}
\end{aligned}
$$

TABLE 2. Reduced expressions for f_j in R.

Because the terms $A(2)/\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1}$ in (2.5) and (2.6) are g-locally acyclic in $\mathcal{D}_{A(2)_*}$, we have cofiber sequences

$$\Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \to g^{-1}\underline{\mathrm{bo}}_{2j} \to \Sigma^{8j+8,1} g^{-1}\underline{\mathrm{bo}}_{j-1} \xrightarrow{\partial'_j} \Sigma^{8j+1,-1} g^{-1}\underline{\mathrm{bo}}_j \tag{7.1}$$

and equivalences

$$g^{-1}\underline{\mathrm{bo}}_{2j+1} \simeq \Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1. \tag{7.2}$$

We therefore get a g-local story completely analogous to the v_2-local story, except much easier, because there are no '$\underline{\mathrm{TMF}_0(3)}$'-terms.

Proposition 7.3. *There is a splitting*

$$g^{-1}\underline{\mathrm{bo}}_1^{\otimes 3} \simeq 2\Sigma^{16,1} g^{-1}\underline{\mathrm{bo}}_1.$$

Proof. This follows the proof of Proposition 5.1, except the situation is simpler because

$$g^{-1}(\underline{\mathrm{bo}}_1^{\otimes 3})^{hC_3} \simeq 0$$

since $g^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3})^{C_3}$ is zero by inspection. □

We also have the following g-local analog of Proposition 2.2, whose proof is identical.

Proposition 7.4. *We have*

$$g^{-1}D\underline{\mathrm{bo}}_1 \simeq \Sigma^{-16,-1} g^{-1}\underline{\mathrm{bo}}_1.$$

Thus we may analyze the decompositions of $g^{-1}\underline{\mathrm{bo}}_j$ by means of generating functions analogous to Section 6. In light of Proposition 7.3, instead of working in the ring R, we work in the ring

$$R' := \mathbb{Z}[s^{\pm}, t^{\pm}, x]/(x^3 = 2t^2 sx).$$

By Proposition 7.4, we may encode g-local Spanier-Whitehead duality by the function

$$\begin{aligned} D : R' &\to R' \\ s &\mapsto s^{-1} \\ t &\mapsto t^{-1} \\ x &\mapsto t^{-2}s^{-1}x \end{aligned}$$

Define elements $f'_j \in R'$ by the same inductive definition (6.3) used to define the elements $f_j \in R$. A simple induction reveals the following.

Lemma 7.5. *The elements $f'_j \in R'$ take the form*

$$f'_j = \begin{cases} \sum_i (a_{i,j} s^i t^j + b_{i,j} s^i t^{j-1} x + c_{i,j} s^i t^{j-2} x^2), & j \text{ even}, \\ \sum_i (b_{i,j} s^i t^{j-1} x + c_{i,j} s^i t^{j-2} x^2), & j \text{ odd}, \end{cases}$$

for $a_{i,j}, b_{i,j}, c_{i,j} \in \mathbb{N}$.

8. The attaching maps ∂_j and ∂'_j

Theorem 8.1. *The attaching maps ∂_j (2.10) and ∂'_j (7.1) are zero for all j.*

Proof. Write the exact sequence (2.5) as a splice of two short exact sequences

$$\begin{tikzcd}
& & 0 \arrow[dr] & & 0 & & \\
& & & K \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & \Sigma^{8j}\underline{\mathrm{bo}}_j \arrow[r] & \underline{\mathrm{bo}}_{2j} \arrow[ur] \arrow[rr] & & A(2)/\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \arrow[r] & \Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1} \arrow[r] & 0
\end{tikzcd}$$

and let

$$\Sigma^{8j}\underline{\mathrm{bo}}_j \to \underline{\mathrm{bo}}_{2j} \to K \xrightarrow{\alpha} \Sigma^{8j+1,-1}\underline{\mathrm{bo}}_j$$

$$\Sigma^{8j+8,1}\underline{\mathrm{bo}}_{j-1} \xrightarrow{\beta} K \to A(2)/\!/A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \to \Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1}$$

be the cofiber sequences in $\mathcal{D}_{A(2)_*}$ induced from these short exact sequences. Then we have the following commutative diagram in $\mathcal{D}_{A(2)_*}$.

$$\begin{tikzcd}
\Sigma^{8j+8,1}v_2^{-1}\underline{\mathrm{bo}}_{j-1} \arrow[r, "\simeq", "v_2^{-1}\beta"'] \arrow[d] \arrow[rr, bend left, "\partial_j"] & v_2^{-1}K \arrow[r, "v_2^{-1}\alpha"'] \arrow[d] & \Sigma^{8j+1,-1}v_2^{-1}\underline{\mathrm{bo}}_j \arrow[d] \\
\Sigma^{8j+8,1}v_2^{-1}g^{-1}\underline{\mathrm{bo}}_{j-1} \arrow[r, "\simeq", "v_2^{-1}g^{-1}\beta"'] & v_2^{-1}g^{-1}K \arrow[r, "v_2^{-1}g^{-1}\alpha"'] & \Sigma^{8j+1,-1}v_2^{-1}g^{-1}\underline{\mathrm{bo}}_j \\
\Sigma^{8j+8,1}g^{-1}\underline{\mathrm{bo}}_{j-1} \arrow[r, "\simeq", "g^{-1}\beta"'] \arrow[u] \arrow[rr, bend right, "\partial'_j"'] & g^{-1}K \arrow[r, "g^{-1}\alpha"'] \arrow[u] & \Sigma^{8j+1,-1}g^{-1}\underline{\mathrm{bo}}_j \arrow[u]
\end{tikzcd}$$

We therefore have

$$g^{-1}\partial_j = v_2^{-1}\partial'_j. \tag{8.2}$$

Now, Assume inductively that ∂_k and ∂'_k are zero for $k < j$. Then for $k < 2j+1$, $v_2^{-1}\underline{\mathrm{bo}}_k$ and $g^{-1}\underline{\mathrm{bo}}_k$ decomposes in $\mathcal{D}_{A(2)_*}$ as a sum of terms corresponding to the terms of f_k and f'_k, respectively. Note that we have

$$\partial_j \in \pi_{7,2}^{A(2)_*}(v_2^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j),$$
$$\partial'_j \in \pi_{7,2}^{A(2)_*}(g^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j).$$

It follows from Lemma 7.5 that

$$D(f'_{j-1}) \cdot f'_j = \sum_i (\alpha_i s^i x + \beta_i s^i t^{-1} x^2)$$

for $\alpha_i, \beta_i \in \mathbb{N}$, and therefore

$$g^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j \simeq \bigoplus_i (\alpha_i \Sigma^{0,i} g^{-1}\underline{\mathrm{bo}}_1 + \beta_i \Sigma^{-8,i} g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}). \tag{8.3}$$

Note that there is a map of rings

$$\phi : R' \to R$$

sending s to s, t to t, and x to x. We have

$$f_k \equiv \phi(f'_k) \mod y.$$

We therefore have

$$D(f_{j-1}) \cdot f_j = \sum_i (\alpha_i s^i x + \beta_i s^i t^{-1} x^2) + \sum_{k,l} \gamma_{k,l} s^k t^l y.$$

It follows that we have
(8.4)
$$v_2^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j \simeq \bigoplus_i (\alpha_i \Sigma^{0,i} v_2^{-1}\underline{\mathrm{bo}}_1 + \beta_i \Sigma^{-8,i} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \oplus \bigoplus_{k,l} \Sigma^{8l,k}\underline{\mathrm{TMF}_0(3)}.$$

Note that

$$\pi_{8m+7,n}^{A(2)_*}(\underline{\mathrm{TMF}_0(3)}) = 0$$

for all n, m, so the the only potential non-zero components of ∂_j under the decomposition (8.4) are the components

$$(\partial_j)_i^{(1)} \in \pi_{7,2-i}(\alpha_i v_2^{-1}\underline{\mathrm{bo}}_1),$$
$$(\partial_j)_i^{(2)} \in \pi_{15,2-i}(\beta_i v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}).$$

Similarly, let

$$(\partial'_j)_i^{(1)} \in \pi_{7,2-i}(\alpha_i g^{-1}\underline{\mathrm{bo}}_1),$$
$$(\partial'_j)_i^{(2)} \in \pi_{15,2-i}(\beta_i g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2})$$

denote the components of ∂'_j under the splitting (8.3).

Note that the splittings (8.3) and (8.4) are compatible under the maps

$$g^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j \to v_2^{-1}g^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j \leftarrow v_2^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j$$

since $g^{-1}\underline{\mathrm{TMF}_0(3)} \simeq 0$, and by (8.2) ∂'_j and ∂_j map to the same element of

$$\pi_{7,2}^{A(2)_*}(v_2^{-1}g^{-1}D(\underline{\mathrm{bo}}_{j-1}) \otimes \underline{\mathrm{bo}}_j).$$

We therefore deduce that under the maps

$$\alpha_i g^{-1}\underline{\mathrm{bo}}_1 \to \alpha_i v_2^{-1}g^{-1}\underline{\mathrm{bo}}_1 \leftarrow \alpha_i v_2^{-1}\underline{\mathrm{bo}}_1,$$
$$\beta_i g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \to \beta_i v_2^{-1}g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2} \leftarrow \beta_i v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}$$

we have

$$v_2^{-1}(\partial'_j)_i^{(1)} = g^{-1}(\partial_j)_i^{(1)},$$
$$v_2^{-1}(\partial'_j)_i^{(2)} = g^{-1}(\partial_j)_i^{(2)}.$$

However, direct inspection of $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1)$ and $\pi_{*,*}^{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2})$ reveals:

- The maps
$$\pi_{7,s}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1) \hookrightarrow \pi_{7,s}^{A(2)_*}(v_2^{-1}g^{-1}\underline{\mathrm{bo}}_1) \hookleftarrow \pi_{7,s}^{A(2)_*}(v_2^{-1}\underline{\mathrm{bo}}_1),$$
$$\pi_{15,s}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \hookrightarrow \pi_{15,s}^{A(2)_*}(v_2^{-1}g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \hookleftarrow \pi_{15,s}^{A(2)_*}(v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2})$$
are injections for all s.
- We have
$$\pi_{7,s}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1) = 0,$$
$$\pi_{15,s}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) = 0$$
for $s \geq 1$.
- We have
$$\pi_{7,s}^{A(2)_*}(v_2^{-1}\underline{\mathrm{bo}}_1) = 0,$$
$$\pi_{15,s}^{A(2)_*}(v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) = 0$$
for $s \leq 1$.

It follows that we must have

$$\begin{aligned}(\partial_j)_i^{(1)} &= 0,\\ (\partial'_j)_i^{(1)} &= 0,\\ (\partial_j)_i^{(2)} &= 0,\\ (\partial'_j)_i^{(2)} &= 0.\end{aligned}$$

□

Corollary 8.5. *We have*

$$g^{-1}\underline{\mathrm{bo}}_{2j} \simeq \Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \oplus \Sigma^{8j+8,1} g^{-1}\underline{\mathrm{bo}}_{j-1}.$$

Therefore, if we write f'_j in the form

$$f'_j = \sum_i (a_{i,j} s^i t^j + b_{i,j} s^i t^{j-1} x + c_{i,j} s^i t^{j-2} x^2)$$

then we have

$$g^{-1}\underline{\mathrm{bo}}_j \;\simeq\; \bigoplus_i (a_{i,j}\Sigma^{8j,i} g^{-1}\mathbb{F}_2 \;\oplus\; b_{i,j}\Sigma^{8(j-1),i} g^{-1}\underline{\mathrm{bo}}_1 \;\oplus\; c_{i,j}\Sigma^{8(j-2),i} g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}).$$

Corollary 8.6. *We have*

$$v_2^{-1}\underline{\mathrm{bo}}_{2j} \simeq \Sigma^{8j} v_2^{-1}\underline{\mathrm{bo}}_j \oplus \Sigma^{8j+8,1} v_2^{-1}\underline{\mathrm{bo}}_{j-1}.$$

Therefore, if we write f_j in the form

$$f_j = \sum_i (a_{i,j} s^i t^j + b_{i,j} s^i t^{j-1} x + c_{i,j} s^i t^{j-2} x^2) + \sum_{k,l} d_{j,k,l} s^k t^l y$$

then we have

$$\begin{aligned} v_2^{-1}\underline{\mathrm{bo}}_j \simeq \bigoplus_i (a_{i,j}\Sigma^{8j,i} v_2^{-1}\mathbb{F}_2 \oplus b_{i,j}\Sigma^{8(j-1),i} v_2^{-1}\underline{\mathrm{bo}}_1 \oplus c_{i,j}\Sigma^{8(j-2),i} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \\ \oplus \bigoplus_{k,l} d_{k,l}\Sigma^{8l,k}\underline{\mathrm{TMF}_0(3)}. \end{aligned}$$

Corollary 8.7. *Consider the element*

$$h := t f_1 w + t^2 f_2 w^2 + t^3 f_3 w^3 \cdots \in R[[w]].$$

Write the coefficient of w^j in h^n as

$$\sum_i (a_{i,j}^{(n)} s^i t^{2j} + b_{i,j}^{(n)} s^i t^{2j-1} x + c_{i,j}^{(n)} s^i t^{2j-2} x^2) + \sum_{j,k,l} d_{k,l}^{(n)} s^k t^l y$$

then the weight $8j$ summand of $v_2^{-1}\overline{\underline{\mathrm{tmf}}}^{\otimes n}$ decomposes as

$$\begin{aligned} \bigoplus_i (a_{i,j}^{(n)}\Sigma^{16j,i} v_2^{-1}\mathbb{F}_2 \oplus b_{i,j}^{(n)}\Sigma^{16j-8,i} v_2^{-1}\underline{\mathrm{bo}}_1 \oplus c_{i,j}^{(n)}\Sigma^{16j-16,i} v_2^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) \\ \oplus \bigoplus_{k,l} d_{j,k,l}^{(n)}\Sigma^{8l,k}\underline{\mathrm{TMF}_0(3)}. \end{aligned}$$

9. Applications to the g-local algebraic tmf-resolution

Consider the quotient Hopf algebra $C_* := \mathbb{F}_2[\zeta_2]/(\zeta_2^4)$ of $A(2)_*$, with

$$\pi_{*,*}^{C_*}(\mathbb{F}_2) = \mathbb{F}_2[v_1, h_{2,1}].$$

The second author, Bobkova, and Thomas computed the P_2^1-Margolis homology of the tmf-resolution, and in the process computed the structure of $A /\!\!/ A(2)_*^{\otimes n}$ as C_*-comodules. From this one can read off the Ext groups

$$h_{2,1}^{-1}\pi_{*,*}^{C_*}(\underline{\mathrm{tmf}}^{\otimes n})$$

(see [BMQ21, Thm. 3.12]).

The groups $h_{2,1}^{-1}\pi_{*,*}^{C_*}$ are closely related to the groups $g^{-1}\pi_{*,*}^{A(2)_*}$. In [BMQ21, Cor. 3.11], it is proven that for $M \in \mathcal{D}_{A(2)_*}$, there is a v_2^8 Bockstein spectral sequence

$$h_{2,1}^{-1}\pi_{*,*}^{C_*}(M) \otimes \mathbb{F}_2[v_2^8] \Rightarrow g^{-1}\pi_{*,*}^{A(2)_*}(M). \tag{9.1}$$

In this section we would like to explain how Corollary 8.5 can be used to compute $g^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{tmf}}^{\otimes n})$. By relating this to [BBT21], we will show that in the case of $M = \underline{\mathrm{tmf}}^{\otimes n}$, the spectral sequence (9.1) collapses (Theorem 9.3).

We follow [BMQ21] in our summary of the results of [BBT21]. The coaction of C_* is encoded in the dual action of the algebra $E[Q_1, P_2^1]$ on $\underline{\mathrm{tmf}}^{\otimes n}$. Define elements

$$x_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{\zeta_{i+3}}_{j} \otimes 1 \otimes \cdots \otimes 1,$$
$$t_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{\zeta_{i+1}^4}_{j} \otimes 1 \otimes \cdots \otimes 1$$

in $\underline{\mathrm{tmf}}^{\otimes n}$.

For an *ordered* set

$$J = ((i_1, j_1), \ldots, (i_k, j_k))$$

of multi-indices, let

$$|J| := k$$

denote the number of pairs of indices it contains. Define linearly independent sets of elements

$$\mathcal{T}_J \subset \underline{\mathrm{tmf}}^{\otimes n}$$

inductively as follows. Define

$$\mathcal{T}_{(i,j)} = \{x_{i,j}\}.$$

For J as above with $|J|$ odd, define

$$\mathcal{T}_{J,(i,j)} = \{z \cdot x_{i,j}\}_{z \in \mathcal{T}_J},$$
$$\mathcal{T}_{J,(i,j),(i',j')} = \{Q_1(z \cdot x_{i,j}) x_{i',j'}\}_{z \in \mathcal{T}_J} \cup \{Q_1(z \cdot x_{i',j'}) x_{i,j}\}_{z \in \mathcal{T}_J}.$$

Let

$$N_J \subset \underline{\mathrm{tmf}}^{\otimes n}$$

denote the $\mathbb{F}_2$-subspace with basis

$$Q_1\mathcal{T}_J := \{Q_1(z)\}_{z \in \mathcal{T}_J}.$$

While the set $\mathcal{T}_J$ depends on the ordering of J, the subspace N_J does not.

Finally, for a set of pairs of indices

$$J = \{(i_1, j_1), \cdots, (i_k, j_k)\}$$

as before, define

$$x_J t_J := x_{i_1,j_1} t_{i_1,j_1} \cdot \cdots \cdot x_{i_k,j_k} t_{i_k,j_k}.$$

The following is can be read off of the computations of [BBT21].

Theorem 9.2 (Bhattacharya-Bobkova-Thomas)**.** *As modules over $\mathbb{F}_2[h_{2,1}^{\pm}, v_1]$, we have*

$$\begin{aligned} h_{2,1}^{-1}\pi_{*,*}^{C_*}(\underline{\mathrm{tmf}}_*^{\otimes n}) = \\ \mathbb{F}_2[h_{2,1}^{\pm}] \otimes \Bigg(\mathbb{F}_2[v_1]\{x_{J'}t_{J'}\}_{J'} \oplus \bigoplus_{|J| \text{ odd}} N_J\{x_{J'}t_{J'}\}_{J\cap J'=\emptyset} \\ \oplus \bigoplus_{|J|\neq 0 \text{ even}} \mathbb{F}_2[v_1]/v_1^2 \otimes N_J\{x_{J'}t_{J'}\}_{J\cap J'=\emptyset} \Bigg) \end{aligned}$$

where J and J' range over the subsets of

$$\{(i,j) \,:\, 1 \le i, 1 \le j \le n\}$$

and v_1 acts trivially on N_J for $|J|$ odd.

We now explain how the equivalences

$$\begin{aligned} g^{-1}\underline{\mathrm{bo}}_{2j} &\simeq \Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \oplus \Sigma^{8j+8,1} g^{-1}\underline{\mathrm{bo}}_{j-1}, \\ g^{-1}\underline{\mathrm{bo}}_{2j+1} &\simeq \Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \end{aligned}$$

are related to Theorem 9.2. This analysis comes from the definitions of the maps of (2.5) and (2.6) in [BHHM08]. For a set J of indices of the form

$$J = \{(i_1, 1), \cdots, (i_k, 1)\},$$

define $J + \Delta$ to be the set

$$J + \Delta = \{(i_1 + 1, 1), \cdots, (i_k + 1, 1)\}.$$

Then the induced maps on homotopy are determined by:

$$\begin{aligned} \pi_{*,*}^{A(2)_*}(\Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j) &\to \pi_{*,*}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_{2j}) \\ N_J\{x_{J'}t_{J'}\} &\mapsto N_{J+\Delta}\{x_{J'+\Delta}t_{J'+\Delta}\} \end{aligned}$$

$$\begin{aligned} \pi_{*,*}^{A(2)_*}(\Sigma^{8j+8,1} g^{-1}\underline{\mathrm{bo}}_{j-1} &\to \pi_{*,*}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_{2j}) \\ N_J\{x_{J'}t_{J'}\} &\mapsto h_{2,1} \cdot N_{J+\Delta}\{x_{1,1}t_{1,1}x_{J'+\Delta}t_{J'+\Delta}\} \end{aligned}$$

$$\begin{aligned} \pi_{*,*}^{A(2)_*}(\Sigma^{8j} g^{-1}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1) &= \pi_{*,*}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_{2j+1}) \\ N_{J\cup\{(1,2)\}}\{x_{J'}t_{J'}\} &\mapsto N_{(J+\Delta)\cup\{(1,1)\}}\{x_{J'+\Delta}t_{J'+\Delta}\}. \end{aligned}$$

We have (with $g = h_{2,1}^4$)

$$\begin{aligned}
\pi_{*,*}^{A(2)_*}(g^{-1}\mathbb{F}_2) &= \mathbb{F}_2[h_{2,1}^{\pm}, v_1, v_2^8],\\
\pi_{*,*}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1) &= \mathbb{F}_2[h_{2,1}^{\pm}, v_1, v_2^8]/(v_1)\{t_{1,1}\},\\
\pi_{*,*}^{A(2)_*}(g^{-1}\underline{\mathrm{bo}}_1^{\otimes 2}) &= \mathbb{F}_2[h_{2,1}^{\pm}, v_1, v_2^8]/(v_1^2)\{Q_1(x_{1,1}x_{1,2})\}.
\end{aligned}$$

Corollary 8.5 therefore implies the following extension of Theorem 9.2.

Theorem 9.3. *As modules over $\mathbb{F}_2[h_{2,1}^{\pm}, v_1, v_2^8]$, we have*

$$\begin{aligned}
&g^{-1}\pi_{*,*}^{A(2)_*}(\underline{\mathrm{tmf}}_*^{\otimes n}) = \\
&\qquad \mathbb{F}_2[h_{2,1}^{\pm}, v_2^8] \otimes \Bigg(\mathbb{F}_2[v_1]\{x_{J'}t_{J'}\}_{J'} \oplus \bigoplus_{|J| \text{ odd}} N_J\{x_{J'}t_{J'}\}_{J\cap J'=\emptyset}\\
&\qquad\qquad \oplus \bigoplus_{|J|\neq 0 \text{ even}} \mathbb{F}_2[v_1]/v_1^2 \otimes N_J\{x_{J'}t_{J'}\}_{J\cap J'=\emptyset}\Bigg)
\end{aligned}$$

where J and J' range over the subsets of

$$\{(i,j) \,:\, 1 \le i, 1 \le j \le n\}$$

and v_1 acts trivially on N_J for $|J|$ odd.

Appendix A. A splitting of $\mathrm{bo}_1^{\wedge 3}$

The v_2-local splitting of Proposition 5.1 comes from a stable splitting of $\mathrm{bo}_1^{\wedge 3}$ induced by an idempotent decomposition of the identity element

$$1 = \mathsf{f}_1 + \mathsf{f}_2 + \mathsf{e} \in \mathbb{Z}_{(2)}[\Sigma_3]$$

as described in Remark A.2. More precisely, if we set

$$F_i := \mathrm{hocolim}\{\mathrm{bo}_1^{\wedge 3} \xrightarrow{\mathsf{f}_i} \mathrm{bo}_1^{\wedge 3} \xrightarrow{\mathsf{f}_i} \dots\}$$

for $i \in \{1, 2\}$ and

$$E := \mathrm{hocolim}\{\mathrm{bo}_1^{\wedge 3} \xrightarrow{\mathsf{e}} \mathrm{bo}_1^{\wedge 3} \xrightarrow{\mathsf{e}} \dots\},$$

using the evident permutation action of Σ_3 on $\mathrm{bo}_1^{\wedge 3}$, then it is easy to see that

$$\mathrm{bo}_1^{\wedge 3} \simeq F_1 \vee F_2 \vee E. \tag{A.1}$$

In fact, F_1, F_2 and E are finite spectra and their mod 2 cohomology as a Steenrod module can be easily computed using the cocommutativity of Steenrod operations and a Künneth isomorphism (see [Rav92, Appendix C]). For the purposes of this paper, we only need their underlying $A(2)$-module structure which we record in the format of a Bruner module definition file [BEM17, Apx. A] (see Figure A.1 and Figure A.2)

Remark A.2. In the group ring $\mathbb{Z}_{(2)}[\Sigma_3]$, the identity element 1 can be written as a sum of idempotent elements

$$\mathsf{f}_1 = \frac{1 + (1\ 2) - (1\ 3) - (1\ 2\ 3)}{3}, \mathsf{f}_2 = \frac{1 + (1\ 3) - (1\ 2) - (1\ 3\ 2)}{3} \text{ and}$$

$$\mathsf{e} = \frac{1 + (1\ 2\ 3) + (1\ 3\ 2)}{3}.$$

```
20

0 2 3 4 6 6 7 7 8 9 9 10 10 11 12 13 13 14 15 16

0 2 1 1
0 3 1 2
0 4 1 3
0 6 1 4
0 7 1 6

1 1 1 2
1 4 1 5
1 5 1 7
1 6 1 8
1 7 1 9

2 4 1 7
2 6 1 10
2 7 1 12

3 2 1 4
3 3 1 6
3 4 1 8
3 5 1 9
3 6 1 12

4 1 1 6
4 4 1 11
4 5 1 13
4 6 1 14
4 7 1 15

5 1 1 7
5 2 1 8
5 3 1 9
5 4 1 12

6 2 1 9
6 4 1 13
6 6 1 16
6 7 1 17

7 2 1 10
7 3 1 12

8 1 1 9
8 2 1 12
8 4 1 14
8 5 1 15
8 6 1 17

9 4 1 15
9 6 1 18
9 7 1 19

10 1 1 12
10 4 1 16
10 5 1 17

11 1 1 13
11 2 1 14
11 3 1 15
11 4 1 17

12 4 1 17
12 6 1 19

13 2 1 16
13 3 1 17
13 4 1 18
13 5 1 19

14 1 1 15
14 2 1 17

15 2 1 18
15 3 1 19

16 1 1 17

17 2 1 19
18 1 1 19
```

FIGURE A.1. The $A(2)$-module structure of $H^*(F_1) \cong H^*(F_2)$ as an input file for Bruner's program

Remark A.3. Note that f_1 and f_2 are conjugates and therefore, $F_1 \simeq F_2$.

Bruner's program is capable of computing the action of $\pi_{*,*}^{A(2)_*}(\mathbb{F}_2)$ on $\pi_{*,*}^{A(2)_*}(M^\vee)$, where $M^\vee$ is the $\mathbb{F}_2$-linear dual of a finite $A(2)$-module M. Therefore, it can be used for verifying the details necessary in the proof of Proposition 5.1 and Proposition 5.2.

Remark A.4. Using Bruner's program and Figure 4.2 one can easily verify

$$v_2^{-1}\pi_{*,*}^{A(2)_*}(H_*(E)) \cong \pi_{*,*}^{A(2)_*}(\Sigma^{24,2}\underline{\mathrm{TMF}_0(3)}).$$

Then by Theorem 4.3 we get $\Sigma^{24,2}\underline{\mathrm{TMF}_0(3)} \simeq v_2^{-1}H_*(E)$ in $\mathcal{D}_{A(2)_*}$.

Remark A.5 (A different proof of Proposition 5.1)**.** Let M_1 denote the first integral Brown-Gitler module. It consists of three $\mathbb{F}_2$-generators $\{x_0, x_2, x_3\}$ where $|x_i| = i$ such that

$$Sq^2(x_0) = x_2 \text{ and } Sq^1(x_2) = x_3.$$

```
24

0 4 6 7 8 10 10 11 11 12 12 13 13 14 14 15 16 17 17 18 18 19 20 21

0 4 1 1
0 6 1 2
0 7 1 3

1 2 1 2
1 3 1 3

2 1 1 3
2 4 2 5 6
2 5 2 7 8

3 4 2 7 8
3 6 2 11 12

4 2 2 5 6
4 3 2 7 8
4 4 2 9 10
4 5 2 11 12
4 6 2 13 14
4 7 1 15

5 1 1 7
5 2 1 10
5 3 2 11 12
5 4 2 13 14
5 5 1 15

6 1 1 8
6 2 1 10
6 3 2 11 12
6 4 2 13 14
6 5 1 15

7 2 1 11
7 3 1 14
7 4 1 15
7 6 2 17 18

8 2 1 12
8 3 1 14
8 4 1 15
8 6 2 17 18

9 2 1 13
9 3 1 15
9 4 1 16
9 5 2 17 18
9 6 2 19 20
9 7 1 21

10 1 2 11 12
10 2 1 14
10 4 1 16
10 5 2 17 18
10 6 2 19 20
10 7 1 21

11 1 1 14
11 4 1 17
11 5 1 20
11 6 1 21

12 1 1 14
12 4 1 18
12 5 1 20
12 6 1 21

13 1 1 15
13 4 1 19
13 5 1 21
13 6 1 22
13 7 1 23

14 4 1 20
14 6 1 22
14 7 1 23

15 2 2 17 18
15 4 1 21
15 6 1 23

16 1 2 17 18
16 2 2 19 20
16 3 1 21
16 4 1 22
16 5 1 23

17 1 1 20
17 2 1 21
17 4 1 23

18 1 1 20
18 2 1 21
18 4 1 23

19 1 1 21
19 2 1 22
19 3 1 23

20 2 1 22
20 3 1 23

21 2 1 23

22 1 1 23
```

FIGURE A.2. The $A(2)$-module structure of $H^*(E)$ as an input file for Bruner's program

It is tedious but straightforward to check that there is a short exact sequence

$$0 \to H^*(\Sigma^{17}\mathrm{bo}_1) \longrightarrow \Sigma^4 A(2)/\!\!/A(1) \otimes M_1 \longrightarrow H^*E \to 0$$

of $A(2)$-modules. This short exact sequence translates into an $\mathcal{D}_{A(2)_*}$-equivalence

$$v_2^{-1}H_*(F_1) \cong H_*(F_2) \simeq \Sigma^{16,1} v_2^{-1}\underline{\mathrm{bo}}_1$$

which, along with Remark A.4 and (A.1), gives yet another proof of Proposition 5.1.

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