BUILDINGS, ELLIPTIC CURVES, AND THE $K(2)$-LOCAL SPHERE

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ABSTRACT. We investigate a dense subgroup Γ of the second Morava stabilizer group given by a certain group of quasi-isogenies of a supersingular elliptic curve in characteristic p. The group Γ acts on the Bruhat-Tits building for $GL_2(\mathbb{Q}_\ell)$ through its action on the ℓ-adic Tate module. This action has finite stabilizers, giving a small resolution for the homotopy fixed point spectrum $(E_2^{h\Gamma})^{hGal}$ by spectra of topological modular forms. Here, E_2 is a version of Morava E-theory and $Gal = Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$.

CONTENTS

1. INTRODUCTION

1.1. **Background.** Fix a prime p. One systematic way of understanding the p-local stable homotopy groups of a finite complex X is to study its chromatic tower, given by the inverse system

$$X_{E(0)} \leftarrow X_{E(1)} \leftarrow X_{E(2)} \leftarrow \cdots .$$

Here $X_{E(n)}$ is Bousfield localization with respect to the Johnson-Wilson spectrum $E(n)$. The chromatic convergence theorem of Hopkins and Ravenel [29] states that this tower converges in the sense that

$$X \simeq \operatorname*{holim}_n X_{E(n)}$$

for all p-local finite complexes X, and that the induced filtration on the homotopy groups of X is exhaustive. The chromatic program for understanding the stable homotopy of X begins with understanding the filtration quotients of this tower.

This is equivalent to studying the localizations $X_{K(n)}$ with respect to Morava K-theory. A very nice summary of this process may be found in the introduction of [15].

We confine our attention to the case $X = S$, the sphere spectrum. Morava [26] developed a method of understanding the layers $S_{K(n)}$, which was strengthened by work of Hopkins-Miller [30], Goerss-Hopkins [16], Devinatz-Hopkins [12], and Davis [9]. We briefly summarize their work. Let E_n be (maximally unramified) Morava E-theory. It is a complex orientable spectrum whose associated formal group is the Lubin-Tate universal deformation of the Honda height n formal group H_n over $\overline{\mathbb{F}}_p$. Our notation is unconventional: E_n is usually taken with respect to the finite field $\mathbb{F}_{p^n}$ instead. We choose to work over $\overline{\mathbb{F}}_p$ because any two height n formal groups are isomorphic over $\overline{\mathbb{F}}_p$ [28, A2.2.11]. Let $\mathbb{S}_n$ denote the Morava stabilizer group $\mathrm{Aut}(H_n)$. Let $\mathbb{G}_n$ denote the larger group of automorphisms which are allowed to act non-trivially on the ground field $\overline{\mathbb{F}}_p$

$$\mathbb{G}_n = \mathrm{Aut}_{/\mathbb{F}_p}(H_n) = \mathbb{S}_n \rtimes Gal.$$

Here Gal is the Galois group $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The spectrum E_n is an E_∞-ring spectrum which is a continuous $\mathbb{G}_n$-spectrum, and there is an equivalence

$$S_{K(n)} \simeq E_n^{h\mathbb{G}_n} \simeq (E_n^{h\mathbb{S}_n})^{hGal}.$$

Computationally, it has proven easier to work with homotopy fixed point spectra E_n^{hF} for *finite* subgroups F of the Morava stabilizer group. For example, when $n = 1$, the p-complete real K-theory spectrum KO_p is equivalent to the homotopy fixed point spectrum $(E_1^{hC_2})^{hGal}$. Choose ℓ to be a topological generator of the group $\mathbb{Z}_p^\times/\{\pm1\}$. The J-theory spectrum is given as the fiber

$$J \to KO_p \xrightarrow{\psi^\ell - 1} KO_p$$

where ψ^ℓ is the ℓth Adams operation. Adams-Baird and Ravenel [5], [27] proved that there is an equivalence $S_{K(1)} \simeq J$. Thus the $K(1)$-local sphere admits a complete description in terms of K-theory.

Goerss, Henn, Mahowald, and Rezk [15] gave a similar decomposition of $S_{K(2)}$ at the prime 3. In their work, certain spectra related to the Hopkins-Miller spectrum of topological modular forms (TMF) played the role that K-theory played in chromatic level 1. This decomposition shed considerable light on the very difficult $K(2)$-local computations of Shimomura and Wang [33]. However, the decomposition was produced by means of obstruction theory and computation, and the attaching maps in their decomposition were not identified explicitly. The decomposition was also specific to the prime 3.

1.2. **A higher analog of the J-theory spectrum.** It is natural to ask if there is an analog of the J-theory spectrum for chromatic level 2 which is built out topological modular forms in a manner similar to the way in which the J-theory spectrum is built out of K-theory. In [2], motivated by [15] and [24], we introduced a spectrum $Q(\ell)$ as the totalization of a semi-cosimplicial spectrum

(1.2.1) $TMF \Rightarrow TMF \times TMF_0(\ell) \Rrightarrow TMF_0(\ell).$

The spectrum TMF is the Hopkins-Miller spectrum of topological modular forms, and the spectrum $TMF_0(\ell)$ is an analogous spectrum associated to the congruence subgroup $\Gamma_0(\ell)$ of $SL_2(\mathbb{Z})$. Diagram 1.2.1 has a very natural *abstract* construction

in terms of moduli of certain diagrams of isogenies of elliptic curves. We refer the reader to [2] for this construction, which relies on unpublished work of Hopkins, Miller, and their collaborators. However, in [2] we also gave a $K(2)$-local construction for the case $p = 3$ and $\ell = 2$ which only used the Goerss-Hopkins-Miller Theorem. In Section 6.1 we extend this $K(2)$-local construction to all primes p and ℓ.

The J-theory spectrum may also be regarded as the homotopy fixed point spectrum

$$J = (E_1^{h \pm \ell^{\mathbb{Z}}})^{hGal}$$

where $\pm \ell^{\mathbb{Z}}$ is the dense subgroup of the Morava stabilizer group $\mathbb{S}_1 \cong \mathbb{Z}_p^{\times}$ generated by ℓ. The dense subgroup $\pm \ell^{\mathbb{Z}}$ is the subgroup of the group of automorphisms of the multiplicative formal group $\widehat{\mathbb{G}}_m$ which is generated by the quasi-isogenies of the multiplicative group $\mathbb{G}_m$ of degree a power of ℓ.

In this paper we aim to give a similar homotopy fixed point construction of the spectrum $Q(\ell)_{K(2)}$. To this end we define a subgroup Γ of $\mathbb{S}_2$ generated by isogenies of a supersingular elliptic curve of degree a power of ℓ. For appropriate choices of ℓ, Tyler Lawson and the author [4] have shown that this subgroup is dense in $\mathbb{S}_2$ (it is dense in an index 2 subgroup if $p = 2$). There is an extension Γ_{Gal} of Γ of the form

$$1 \to \Gamma \to \Gamma_{Gal} \to \sigma^{\mathbb{Z}} \to 1$$

where $\sigma^{\mathbb{Z}}$ is the dense subgroup of Gal generated by the Frobenius. The group Γ_{Gal} is dense in $\mathbb{G}_2$ (respectively, an index 2 subgroup if $p = 2$). Let $E(\Gamma)$ denote the homotopy fixed point spectrum

$$E(\Gamma) = E_2^{h\Gamma_{Gal}}.$$

Because Γ_{Gal} is dense in $\mathbb{G}_2$, one expects that $E(\Gamma)$ is closely related to the the $K(2)$-local sphere $S_{K(2)} = E_2^{h\mathbb{G}_2}$. The precise conjecture is explained in Section 1.6.

We shall discuss the following.

(1) The group Γ acts on the Bruhat-Tits building for $GL_2(\mathbb{Q}_\ell)$ with finite stabilizers.
(2) This action gives a presentation of the group Γ in terms of the category of supersingular curves over $\overline{\mathbb{F}}_p$.
(3) The action of Γ on the building induces a decomposition of $E(\Gamma)$ in terms of $K(2)$-local topological modular forms.
(4) This decomposition induces an equivalence $Q(\ell)_{K(2)} \simeq E(\Gamma)$.

The remainder of the introduction is devoted to a more detailed discussion the results and organization of this paper.

Remark 1.2.2. There is work by other authors which bears some similarity to the contents of this paper.

- Gorbounov, Mahowald, and Symonds [17] produced dense amalgamated products of finite subgroups of the Morava stabilizer group $\mathbb{S}_{p-1}$. Our dense subgroups appear to differ from theirs in the overlapping case of $p = 3$.
- In the case of $p = 3$, the computations of Gorbounov, Siegel, and Symonds [18] reflect algebraically an analog of Conjecture 1.6.1.
- Andrew Baker [1] has shown that for $p > 3$ the E_2-term of the ANSS for $S_{K(2)}$ can be computed as the cohomology of the *p-completed* groupoid of supersingular elliptic curves and isogenies.

1.3. **The subgroup** Γ. For the remainder of this paper assume we are given a fixed supersingular elliptic curve C over $\overline{\mathbb{F}}_p$ and a prime ℓ coprime to p. Our work will turn out to be independent of the choice of supersingular curve — the choice is tantamount to choosing a basepoint in a connected category supersingular curves. For convenience we shall insist that C admits a definition over $\mathbb{F}_p$ (for every prime p such a supersingular curve exists [36]).

Since C is supersingular, the formal completion $C^\wedge$ of C at the identity is isomorphic to the Honda height 2 formal group H_2. One may regard $C^\wedge$ as the p-divisible group $C[p^\infty]$.

Our intention is to study the simultaneous action of the endomorphism ring $\mathrm{End}(C)$ on the p-torsion and ℓ-torsion of C. Let $\Gamma \subset \mathrm{End}(C) \otimes \mathbb{Q}$ be the group of quasi-isogenies of C with degree equal to a power of ℓ. Then we have the following diagram.

(1.3.1)

Tate [35],[37] proved that the top inclusions are actually the ℓ and p-completions of the endomorphism ring, respectively. In [4], we proved the following theorem.

Theorem 1.3.2 (Behrens-Lawson [4]). Let ℓ be a topological generator of $\mathbb{Z}_p^\times$ ($\mathbb{Z}_2^\times/\{\pm 1\}$ if $p = 2$). For $p > 2$, the group Γ is dense in $\mathbb{S}_2$. For $p = 2$, the group Γ is dense in the index 2 subgroup $\widetilde{\mathbb{S}}_2$ which is the kernel of the composite

$$\mathbb{S}_2 \xrightarrow{N} \mathbb{Z}_2^\times \to (\mathbb{Z}/8^\times)/\{1, \ell\}.$$

1.4. **The building.** We shall denote X^{ss} to be the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. Let $X_0^{ss}(\ell)$ be the set of isomorphism classes of pairs (C', H) where C' is a supersingular elliptic curve and H is a $\Gamma_0(\ell)$-structure (a subgroup of order ℓ contained in $C'(\overline{\mathbb{F}}_p)$). Given a pair $(C', H) \in X_0^{ss}(\ell)$, let $\mathrm{Aut}(C', H)$ be the group of automorphisms ϕ of C' such that $\phi(H) = H$.

The group Γ naturally acts on the ℓ-adic Tate module $V_\ell(C)$, giving an inclusion into the group $GL_2(\mathbb{Q}_\ell)$. Let $\mathcal{J}'$ be the Bruhat-Tits building for $GL_2(\mathbb{Q}_\ell)$. $\mathcal{J}'$ is a 2-dimensional contractible simplicial complex on which $GL_2(\mathbb{Q}_\ell)$ acts. The induced Γ action on $\mathcal{J}'$ has finite stabilizers, which are given naturally by certain groups automorphisms of supersingular elliptic curves.

Theorem (Theorem 3.4.4). The building $\mathcal{J}'$ is Γ-equivariantly homeomorphic to the geometric realization of a Γ-equivariant semi-simplicial complex $\mathcal{J}'_\bullet$. The simplices of $\mathcal{J}'_\bullet$ are given as follows:

$$\mathcal{J}'_0 = \coprod_{C' \in X^{ss}} \Gamma / \operatorname{Aut}(C'),$$

$$\mathcal{J}'_1 = \coprod_{(C',H) \in X_0^{ss}(\ell)} \Gamma / \operatorname{Aut}(C',H) \amalg \coprod_{C' \in X^{ss}} \Gamma / \operatorname{Aut}(C'),$$

$$\mathcal{J}'_2 = \coprod_{(C',H) \in X_0^{ss}(\ell)} \Gamma / \operatorname{Aut}(C',H).$$

1.5. **A resolution of** $E(\Gamma)$**.** The semi-simplicial complex of Theorem 3.4.4 gives rise to the following semi-cosimplicial construction.

Proposition (Proposition 6.2.6). There is a semi-cosimplicial E_∞-ring spectrum of the form

$$\prod_{C' \in X^{ss}} E_2^{h \operatorname{Aut}(C')} \quad \Rrightarrow \quad \begin{matrix} \prod_{C' \in X^{ss}} E_2^{h \operatorname{Aut}(C')} \\ \times \\ \prod_{(C',H) \in X_0^{ss}(\ell)} E_2^{h \operatorname{Aut}(C',H)} \end{matrix} \quad \Rrightarrow \quad \prod_{(C',H) \in X_0^{ss}(\ell)} E_2^{h \operatorname{Aut}(C',H)}$$

which totalizes to give the homotopy fixed point spectrum $E_2^{h\Gamma}$.

As we shall explain in Section 5, the $K(2)$-localizations of TMF and $TMF_0(\ell)$ are given as products of the following homotopy fixed point spectra:

$$TMF_{K(2)} \simeq \left(\prod_{C' \in X^{ss}} E_2^{h \operatorname{Aut}(C')} \right)^{hGal},$$

$$TMF_0(\ell)_{K(2)} \simeq \left(\prod_{(C',H) \in X_0^{ss}(\ell)} E_2^{h \operatorname{Aut}(C',H)} \right)^{hGal}.$$

These constructions, due to Hopkins, Miller, and their collaborators, have not yet appeared in the literature. Section 5 may be regarded as a self-contained construction of the $K(2)$-local versions of these spectra of topological modular forms.

It turns out, upon taking Galois homotopy fixed points, that the semi-cosimplicial spectrum given by Proposition 6.2.6 is the $K(2)$-localization of the semi-cosimplicial spectrum of Diagram (1.2.1) defining $Q(\ell)$. We therefore have the following theorem.

Theorem (Theorem 6.2.1). The spectra $Q(\ell)_{K(2)}$ and $E(\Gamma)$ are naturally equivalent.

We also produce a decomposition of $E_2^{h\Gamma_{Gal}^1}$ (Theorem 6.3.1), where Γ_{Gal}^1 is the subgroup of Γ consisting of elements of norm 1.

1.6. **Relation to** $K(2)$**-local sphere.** While the spectrum J is equivalent to the $K(1)$-local sphere, it appears that the $K(2)$-local sphere is built out of the spectrum $Q(\ell)_{K(2)}$ and a spectrum dual to $Q(\ell)_{K(2)}$. More precisely, we make the following conjecture.

Conjecture 1.6.1. Let ℓ be a generator of $\mathbb{Z}_p^\times$ (respectively $\mathbb{Z}_2^\times/\{\pm 1\}$ for $p = 2$). Then if p is odd, the sequence

$$(1.6.2) \qquad\qquad D_{K(2)}Q(\ell) \xrightarrow{D\eta} S_{K(2)} \xrightarrow{\eta} Q(\ell)_{K(2)}$$

is a cofiber sequence. Here η is the $K(2)$-localization of the unit of the ring spectrum $Q(\ell)$, and $D_{K(2)}$ denotes the Spanier-Whitehead dual in the $K(2)$-local category. At the prime 2, let $\widetilde{S}$ denote the homotopy fixed point spectrum $E_2^{h\widetilde{\mathbb{G}}_2}$. Here, $\widetilde{\mathbb{G}}_2 = \widetilde{\mathbb{S}}_2 \rtimes Gal$ is an index 2 subgroup of $\mathbb{G}_2$, where $\widetilde{\mathbb{S}}_2$ is the group of Theorem 1.3.2. Then there is a cofiber sequence

$$D_{\widetilde{S},K(2)}Q(\ell) \xrightarrow{D\eta} \widetilde{S} \xrightarrow{\eta} Q(\ell)_{K(2)}$$

where $D_{\widetilde{S},K(2)}$ denotes the Spanier-Whitehead dual in the category of $K(2)$-local $\widetilde{S}$-modules.

Remark 1.6.3. There is an equivalence

$$S_{K(2)} \simeq \widetilde{S}^{hC_2}.$$

The generator of the group C_2 lifts to a torsion-free element of the group $\mathbb{G}_2$. Therefore, the $K(2)$-local sphere at the prime 2 differs mildly from the spectrum $\widetilde{S}$.

Remark 1.6.4. Conjecture 1.6.1 hypothesizes that the sequence (1.6.2) *extends* to a cofiber sequence. There are possibly many different extensions, and the lack of a natural candidate represents a major gap in our understanding of $K(2)$-local homotopy theory.

The intuition that something like Conjecture 1.6.1 should be true is due to Mark Mahowald. In [2], we proved Conjecture 1.6.1 in the case $p = 3$ and $\ell = 2$. The author intends to combine Theorem 1.3.2 with Theorem 6.2.1 to prove a version of Conjecture 1.6.1 for $p > 3$ in a future paper.

1.7. **Organization of the paper.** In Section 2, we describe the ring of endomorphisms of the supersingular curve C, and describe its action on the formal group of C and the ℓ-adic Tate module of C. We define the group Γ and show that it is may be viewed as an ℓ-arithmetic group associated to a form of GL_2. We also describe an SL_2-variant, and define an associated dense subgroup Γ^1 of the norm 1 subgroup $\mathbb{S}l_2 \subset \mathbb{S}_2$.

In Section 3, we introduce the building $\mathcal{J}'$ for $GL_2(\mathbb{Q}_\ell)$ from the point of view of $\mathbb{Z}_\ell$-lattices in $\mathbb{Q}_\ell^2$. We then translate this description to one in terms of subgroups of C using the Weil pairing. We immediately deduce the Γ-equivariant structure of $\mathcal{J}'$.

In Section 4, we introduce the building (tree) for $SL_2(\mathbb{Q}_\ell)$ and run a similar analysis to that of Section 3, with the group Γ replaced by its norm 1 counterpart Γ^1. We deduce amalgamation formulas for Γ^1 using Bass-Serre theory.

We begin Section 5 with a review of the Goerss-Hopkins-Miller functor, and a technical discussion of the homotopy fixed point construction of Devinatz and Hopkins. We then give an exposition of the construction of the spectra $TMF_{K(2)}$ and $TMF_0(\ell)_{K(2)}$ of Goerss, Hopkins, Miller and their collaborators.

In Section 6, we give a $K(2)$-local construction of the spectrum $Q(\ell)$. We then show that this spectrum is naturally equivalent to the spectrum $E(\Gamma)$. We end by describing a variant where the group Γ is replaced by the norm 1 subgroup Γ^1.

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2. The ring of endomorphisms of C

Let $\mathrm{End}(C)$ be the ring of endomorphisms of C defined over the algebraic closure $\overline{\mathbb{F}}_p$. Let D be the ring of quasi-isogenies

$$D = \mathrm{End}^0(C) = \mathrm{End}(C) \otimes \mathbb{Q}.$$

Because C is supersingular, D is the quaternion algebra over $\mathbb{Q}$ ramified at p and ∞ [34]. The subring $\mathrm{End}(C) \subset D$ is a maximal order. For v a valuation, let $D_v = D \otimes_{\mathbb{Q}} \mathbb{Q}_v$ be the completion of D at v. We shall recall in this section how the elliptic curve C gives a very explicit description of these local algebras.

2.1. **The ℓ-torsion of C.** For any prime ℓ different from p, there is a non-canonical isomorphism of groups

$$C[\ell^\infty] \cong \mathbb{Z}/\ell^\infty \times \mathbb{Z}/\ell^\infty.$$

Let $T_\ell(C)$ be the ℓ-adic Tate module. It is the inverse limit of the inverse system

$$C[\ell] \xleftarrow{[\ell]} C[\ell^2] \xleftarrow{[\ell]} C[\ell^3] \xleftarrow{[\ell]} \cdots.$$

The $\mathbb{Z}_\ell$-module $T_\ell(C)$ is free of rank 2. Since every endomorphism of C restricts to an endomorphism of $C[\ell^k]$, we see that $T_\ell(C)$ is a module over the ring $\mathrm{End}(C)$. We recall the following fundamental theorem of Tate (the case where A is an elliptic curve, as well as Corollary 2.1.2, may be deduced from the work of Deuring [10]).

Theorem 2.1.1 (Tate [35]). *Let A is an abelian variety over the finite field $\mathbb{F}_q$. Then the natural map*

$$\mathrm{End}(A) \otimes \mathbb{Z}_\ell \to \mathrm{End}_{\mathbb{Z}_\ell[\mathrm{Frob}_q^{rel}]}(T_\ell(A))$$

is an isomorphism, where Frob_q^{rel} is the endomorphism induced by the qth relative Frobenius.

For supersingular elliptic curves C', some power of the relative Frobenius will lie in the center of $\mathrm{End}(T_\ell(C'))$, so we have the following corollary.

Corollary 2.1.2. *Let C' be a supersingular elliptic curve over $\overline{\mathbb{F}}_p$. Then the natural map*

$$\mathrm{End}(C') \otimes \mathbb{Z}_\ell \to \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(C'))$$

is an isomorphism.

Corollary 2.1.3. *The algebra D_ℓ is split (isomorphic to $M_2(\mathbb{Q}_\ell)$).*

The Tate module may be equated with the Pontryagin dual of the ℓ-torsion subgroup $C[\ell^\infty]$ as follows.

Proposition 2.1.4. The Weil pairing induces a Galois equivariant isomorphism

$$\widetilde{e} : T_\ell(C) \to \mathrm{Hom}(C[\ell^\infty], \mu_{\ell^\infty}) = C[\ell^\infty]^*$$

where μ_{ℓ^∞} is the ℓ-torsion in the multiplicative group $\overline{\mathbb{F}}_p^\times$, and the Galois group acts by conjugation on $\mathrm{Hom}(C[\ell^\infty], \mu_{\ell^\infty})$.

Proof. Recall that the Weil pairing is a bilinear Galois equivariant non-degenerate pairing

$$e_{\ell^k} : C[\ell^k] \times C[\ell^k] \to \mu_{\ell^k}.$$

Non-degeneracy implies that the adjoint homomorphism is an isomorphism.

$$\widetilde{e}_{\ell^k} : C[\ell^k] \xrightarrow{\cong} \mathrm{Hom}(C[\ell^k], \mu_{\ell^k})$$
$$x \mapsto e_{\ell^k}(x, -)$$

One of the properties of the Weil pairing is that the following diagram commutes [34, III.8.1]

$$
\begin{array}{ccc}
C[\ell^{k+1}] & \xrightarrow{\widetilde{e}_{\ell^{k+1}}} & \mathrm{Hom}(C[\ell^{k+1}], \mu_{\ell^{k+1}}) \\
\downarrow{\scriptstyle [\ell]} & & \downarrow{\scriptstyle \iota^*} \\
C[\ell^k] & \xrightarrow{\widetilde{e}_{\ell^k}} & \mathrm{Hom}(C[\ell^k], \mu_{\ell^k})
\end{array}
$$

where $\iota : C[\ell^k] \hookrightarrow C[\ell^{k+1}]$ is the inclusion. The isomorphism $\widetilde{e}$ is the composite

$$
\begin{aligned}
T_\ell(C) = \lim_k C[\ell^k] & \\
\xrightarrow{\cong} & \lim_k \mathrm{Hom}(C[\ell^k], \mu_{\ell^k}) \\
\xrightarrow{\cong} & \lim_k \mathrm{Hom}(C[\ell^k], \mu_{\ell^\infty}) \\
\xrightarrow{\cong} & \mathrm{Hom}(\mathrm{colim}_k C[\ell^k], \mu_{\ell^\infty}) \\
\xrightarrow{\cong} & \mathrm{Hom}(C[\ell^\infty], \mu_{\ell^\infty}).
\end{aligned}
$$

$\square$

The isomorphism $\widetilde{e}$ induces an $\mathrm{End}(C)$-module structure on $\mathrm{Hom}(C[\ell^\infty], \mu_{\ell^\infty})$. This action is given explicitly in the following lemma.

Lemma 2.1.5. Let α be an element of $C[\ell^\infty]^* = \mathrm{Hom}(C[\ell^\infty], \mu_{\ell^\infty})$, and let ϕ be an endomorphism of C. Then the action of ϕ on α is given by pre-composition

$$\phi \cdot \alpha = \alpha \circ \widehat{\phi}$$

where $\widehat{\phi}$ is the dual isogeny.

Proof. This is immediate from the following ajointness property of the Weil pairing [34, III.8.2]. For x and y in $C[\ell^k]$, we have

$$e_{\ell^k}(\phi(x), y) = e_{\ell^k}(x, \widehat{\phi}(y)).$$

$\square$

2.2. **The p-torsion of C.** Because C is supersingular, it has no non-trivial p-torsion points. The p-divisible group $C[p^\infty]$ is entirely formal, meaning that it coincides with the height 2 formal group $C^\wedge$.

The endomorphism ring of $C^\wedge$ is the maximal order of the $\mathbb{Q}_p$-division algebra $D_{p,1/2}$ of Hasse invariant $1/2$. The following theorem is due to Tate.

Theorem 2.2.1 (Tate [37]). The natural map

$$\mathrm{End}(C) \otimes \mathbb{Z}_p \to \mathrm{End}(C[p^\infty]) = \mathrm{End}(C^\wedge)$$

is an isomorphism.

Corollary 2.2.2. The algebra D_p is non-split (isomorphic to $D_{p,1/2}$).

Remark 2.2.3. The fundamental exact sequence of class field theory implies that the local invariants of D must add to zero. Therefore, the quaternion algebra D must ramify at infinity, giving an isomorphism

$$D_\infty \cong \mathbb{H}.$$

2.3. **The reduced norm.** Let R be a ring. Consider the degree map

$$\deg : \mathrm{End}(C) \to \mathbb{Z}.$$

If we choose an additive basis of $\mathrm{End}(C)$, then the degree map is expressed by a degree 2 polynomial in 4 variables. The degree map extends multiplicatively to a reduced norm

$$N_R = \deg \otimes R : \mathrm{End}(C) \otimes R \to R.$$

In particular, $N_\mathbb{Q}$ coincides with the reduced norm of the quaternion algebra D.

2.4. **The group scheme G.** The various groups which appear in Diagram (1.3.1) are conveniently given as the R-points of an affine group scheme G for various R. Define G to be the scheme whose R-points are given by

$$\begin{aligned}
G(R) &= (\mathrm{End}(C) \otimes R)^\times \\
&= \{x \in \mathrm{End}(C) \otimes R \;:\; N_R(x) \in R^\times\}.
\end{aligned}$$

This functor is represented by an affine scheme because $\mathrm{End}(C)$ is free abelian and the reduced norm is given by a polynomial.

Corollary 2.1.2 and Theorem 2.2.1 identify the R points of G for various R.

Proposition 2.4.1. We have the following values of the functor $G(-)$ where ℓ is prime to p:

$$\begin{aligned}
G(\mathbb{Z}) &= \mathrm{Aut}(C), \\
G(\mathbb{Q}) &= D^\times, \\
G(\mathbb{Z}_\ell) &= \mathrm{Aut}(C[\ell^\infty]) = GL(T_\ell(C)) \cong GL_2(\mathbb{Z}_\ell), \\
G(\mathbb{Q}_\ell) &= GL(T_\ell(C) \otimes \mathbb{Q}) \cong GL_2(\mathbb{Q}_\ell), \\
G(\mathbb{Z}_p) &= \mathrm{Aut}(C^\wedge) = \mathbb{S}_2, \\
G(\mathbb{Q}_p) &= D^\times_{p,1/2}, \\
G(\mathbb{R}) &= \mathbb{H}^\times.
\end{aligned}$$

2.5. **The group Γ.** We now fix ℓ to be a topological generator of the group $\mathbb{Z}_p^\times$ (respectively $\mathbb{Z}_2^\times/\{\pm 1\}$). Define $\mathrm{End}_\ell(C)$ to be the monoid of endomorphisms of C with degree a power of ℓ. Let Γ be the group completion of this monoid.

Lemma 2.5.1. The group Γ is given by inverting the element $[\ell]$ of the monoid $\mathrm{End}_\ell(C)$. That is to say, the natural map

$$\mathrm{End}_\ell(C)[\ell^{-1}] \to \Gamma$$

is an isomorphism.

Proof. We simply need to show that $\mathrm{End}_\ell(C)[\ell^{-1}]$ contains inverses for every $\phi \in \mathrm{End}_\ell(C)$. If ϕ has degree ℓ^k, then the dual isogeny $\widehat{\phi}$ has the property [34, III.6.2]

$$\phi\widehat{\phi} = \widehat{\phi}\phi = [\ell^k].$$

Therefore, the element $\ell^{-k} \cdot \widehat{\phi} \in \mathrm{End}_\ell(C)[\ell^{-1}]$ is an inverse for ϕ. $\qquad\square$

The group Γ is therefore the group of quasi-isogenies of C with degree a power of ℓ. Alternatively, Γ is given by G as

$$\Gamma = G(\mathbb{Z}[1/\ell]).$$

There are inclusions

$$\Gamma \hookrightarrow G(\mathbb{Q}_\ell) \cong GL_2(\mathbb{Q}_\ell),$$
$$\Gamma \hookrightarrow G(\mathbb{Z}_p) = \mathbb{S}_2$$

induced by the inclusions of the ring $\mathbb{Z}[1/\ell]$ into $\mathbb{Q}_\ell$ and $\mathbb{Z}_p$. Theorem 1.3.2 says that for ℓ chosen as above, Γ is dense in $\mathbb{S}_2$ (respectively $\widetilde{\mathbb{S}}_2$ for $p = 2$).

2.6. **The kernel of the reduced norm.** Define the affine group scheme G^1 to be the kernel of reduced norm

$$G^1 \to G \xrightarrow{N} \mathbb{G}_m.$$

The R-points of G^1 are given by

$$G^1(R) = \{x \in \mathrm{End}(C) \otimes R \;:\; N_R(x) = 1\}.$$

The $\mathbb{Z}_p$-points give the closed subgroup

$$G^1(\mathbb{Z}_p) = \mathbb{S}l_2$$

of $\mathbb{S}_2$. We warn the reader that this group differs from the group $\mathbb{S}_2^1$ of [15] and [2] in that we have not projected out the Teichmüller lift of $\mathbb{F}_p^\times$ in $\mathbb{Z}_p^\times$. There is therefore a short exact sequence

$$1 \to \mathbb{S}l_2 \to \mathbb{S}_2^1 \to \mathbb{F}_p^\times \to 1.$$

We define Γ^1 to be the group

$$\Gamma^1 = G^1(\mathbb{Z}[1/\ell]).$$

The group Γ^1 is dense in $\mathbb{S}l_2$ [4].

2.7. **Extending by the Galois group.** Let $Gal \cong \widehat{\mathbb{Z}}$ be the Galois group of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$. It is generated by the pth power Frobenius

$$\sigma : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p.$$

In this section we will introduce compatible actions of Gal on all of our endomorphism rings.

Recall that given a scheme X over $\overline{\mathbb{F}}_p$, there are three different Frobenius morphisms, given by the following diagram

where the scheme $X^{(p)}$ is the pullback of X over σ^*. The morphism Frob_p^{rel} is the *relative Frobenius*, and Frob_p^{tot} is the *total Frobenius*. If $X = Y \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, for a scheme Y over $\mathbb{F}_p$, then there is a canonical isomorphism $X \cong X^{(p)}$. In this case, Frob_p is an automorphism of X that covers the automorphism σ of $\overline{\mathbb{F}}_p$.

For each $C' \in X^{ss}$, let $\sigma_* C' \in X^{ss}$ be the target of the map Frob_p whose source is C':

$$\mathrm{Frob}_p : C' \to \sigma_* C'.$$

Since the curve C was assumed to be defined over $\mathbb{F}_p$, we have $\sigma_* C = C$ and Frob_p takes the form

$$\mathrm{Frob}_p : C \to C.$$

For each $\phi \in \mathrm{End}(C)$, the Frobenius σ acts on ϕ by

$$\sigma_* \phi = \mathrm{Frob}_p \, \phi \, \mathrm{Frob}_p^{-1} \in \mathrm{End}(C).$$

Now if ϕ arises from an isogeny defined over $\mathbb{F}_{p^r}$, then we have $(\sigma_*)^r \phi = \phi$. We conclude that we get an induced *continuous* action of Gal on $\mathrm{End}(C)$ by ring homomorphisms. (To be precise, this really should be regarded as an action of Gal^{op}, since iterates of Frob_p covers the action of Gal^{op} on $\mathrm{spec}(\overline{\mathbb{F}}_p)$, but since Gal is abelian, we will ignore this minor point.)

Define $\mathrm{End}_{/\mathbb{F}_p}(C)$ to be the completed twisted group ring

$$\mathrm{End}_{/\mathbb{F}_p}(C) = \mathrm{End}(C)[[Gal]]$$
$$= \lim_r \mathrm{End}(C)[Gal(\mathbb{F}_{p^r}/\mathbb{F}_p)].$$

The ring $\mathrm{End}_{/\mathbb{F}_p}(C)$ consists of endomorphisms of C which do not cover the identity on $\overline{\mathbb{F}}_p$.

The automorphism $\mathrm{Frob}_p : C \to C$ does *not* induce a map on $\overline{\mathbb{F}}_p$-points, because it is not a morphism of schemes over $\overline{\mathbb{F}}_p$. The relative Frobenius is not an automorphism of schemes (since it is not invertible), but it does induce an automorphism on $\overline{\mathbb{F}}_p$-points

$$\mathrm{Frob}_p^{rel} : C(\overline{\mathbb{F}}_p) \to C(\overline{\mathbb{F}}_p).$$

The following lemma is easily proven using local coordinates.

Lemma 2.7.1. Let ϕ be an element of $\mathrm{End}(C)$. Then, on $\overline{\mathbb{F}}_p$-points, the endomorphism $\sigma_*\phi$ is given by the composite

$$\sigma_*\phi : C(\overline{\mathbb{F}}_p) \xrightarrow{(\mathrm{Frob}_p^{rel})^{-1}} C(\overline{\mathbb{F}}_p) \xrightarrow{\phi} C(\overline{\mathbb{F}}_p) \xrightarrow{\mathrm{Frob}_p^{rel}} C(\overline{\mathbb{F}}_p).$$

Let $\sigma^{\mathbb{Z}}$ be the dense subgroup of Gal generated by σ. The action of $\sigma^{\mathbb{Z}}$ on $\mathrm{End}(C)$ by ring automorphisms induces an action of $\sigma^{\mathbb{Z}}$ on Γ by group automorphisms. Since the norm map is invariant under this action, the action of Gal restricts to the norm 1 subgroup Γ^1. These actions give rise to extensions

$$\Gamma_{Gal} = \Gamma \rtimes \sigma^{\mathbb{Z}},$$
$$\Gamma_{Gal}^1 = \Gamma^1 \rtimes \sigma^{\mathbb{Z}}.$$

We have containments

$$\Gamma_{Gal}^1 \subset \Gamma_{Gal} \subset (\mathrm{End}_{/\mathbb{F}_p}(C)[1/\ell])^{\times}.$$

In the last containment, the element σ of Gal gets mapped to the automorphism $\mathrm{Frob}_p \in \mathrm{End}_{/\mathbb{F}_p}(C)$.

The Tate module $T_\ell(C) = \lim_n C[\ell^n]$ inherits a Galois action through the action of Frob_p^{rel}, and conjugation by Frob_p^{rel} induces a Galois action on $\mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(C))$. Lemma 2.7.1 implies the following corollary.

Corollary 2.7.2. The natural map

$$\mathrm{End}(C) \to \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(C))$$

is Galois equivariant.

In a manner completely analogous to the case of $\mathrm{End}(C)$, we may define an action of Gal on $\mathrm{End}(\widehat{C})$ by conjugation with the automorphism Frob_p. The extended Morava stabilizer group is defined by this action:

$$\mathbb{G}_2 = \mathbb{S}_2 \rtimes Gal = \mathrm{Aut}_{/\mathbb{F}_p}(\widehat{C}).$$

The following lemma is clear.

Lemma 2.7.3. The natural map

$$\mathrm{End}(C) \to \mathrm{End}(\widehat{C})$$

is Galois equivariant.

Thus the inclusion $\Gamma \hookrightarrow \mathbb{S}_2$ extends to an inclusion

$$\Gamma_{Gal} \hookrightarrow \mathbb{G}_2.$$

Theorem 1.3.2 implies the following proposition.

Proposition 2.7.4. The group Γ_{Gal} is dense in $\mathbb{G}_2$ (respectively, an index 2 subgroup if $p = 2$).

3. The building for $GL_2(\mathbb{Q}_\ell)$

3.1. **Construction using lattices.** Let V be a $\mathbb{Q}_\ell$ vector space of dimension 2. The Bruhat-Tits building for $GL(V)$ is a contractible 2-dimensional simplicial complex $\mathcal{J}'(V)$ on which $GL(V)$ naturally acts [7].

A lattice L of V is a rank 2 free $\mathbb{Z}_\ell$-submodule such that the $V = \mathbb{Q} \otimes L$. The complex $\mathcal{J}' = \mathcal{J}'(V)$ is the geometric realization of a semisimplicial set

$$\mathcal{J}_0' \Leftarrow \mathcal{J}_1' \Lleftarrow \mathcal{J}_2'$$

where the sets $\mathcal{J}_i'$ are given as the following sets of flags of lattices in V.

$$\mathcal{J}_0' = \{L_0 \ : \ L_0 \text{ a lattice in } V\},$$
$$\mathcal{J}_1' = \{L_0 < L_1 \ : \ L_1/L_0 \cong \mathbb{Z}/\ell \text{ or } \mathbb{Z}/\ell \times \mathbb{Z}/\ell\},$$
$$\mathcal{J}_2' = \{L_0 < L_1 < L_2 \ : \ L_1/L_0 \cong \mathbb{Z}/\ell \text{ and } L_2/L_0 \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell\}.$$

The ith face maps are given by deleting the ith terms of the flags. This semisimplicial set is $GL(V)$ equivariant with the group acting by permuting the flags.

We give a description of the underlying topological space of $\mathcal{J}'$. Let $\mathcal{J}$ be the $\ell + 1$-regular tree (the building for $SL(V)$). It is the infinite tree where every vertex has valence $\ell + 1$.

Proposition 3.1.1 (See, for instance, [6]). The building $\mathcal{J}'$ is homeomorphic to $\mathcal{J} \times \mathbb{R}$. In particular, it is contractible.

3.2. **Lattices and virtual subgroups.** We now fix V to be the 2-dimensional $\mathbb{Q}_\ell$-vector space

$$V = V_\ell(C) = T_\ell(C) \otimes \mathbb{Q}.$$

We shall give a different perspective of the building $\mathcal{J}'(V_\ell(C))$ which is more convenient for understanding the action of the subgroup Γ of $GL(V_\ell(C))$. The lattices of the previous section will be replaced with certain generalized subgroups of the group $C[\ell^\infty]$, which we shall refer to as virtual subgroups. These should be thought of as the kernels of certain quasi-isogenies. In this section we define the set of virtual subgroups of $C[\ell^\infty]$, and show that they are in bijective correspondence with the set of lattices in $V_\ell(C)$.

Let $\mathrm{Sub}_\ell(C)$ be the set of finite subgroups of $C[\ell^\infty]$. The set $\mathrm{Sub}_\ell(C)$ carries a natural action of the monoid $\mathrm{End}_\ell(C)$ (Section 2.5). Namely, given an endomorphism ϕ in $\mathrm{End}_\ell(C)$, let ϕ act on $\mathrm{Sub}_\ell(C)$ by

$$\phi : H \mapsto \widehat{\phi}^{-1}(H).$$

The inverse image $\widehat{\phi}^{-1}(H)$ is again a finite subgroup of $C[\ell^\infty]$. The order of the kernel of $\widehat{\phi}$ is the degree

$$\deg(\widehat{\phi}) = \deg(\phi)$$

which is a power of ℓ.

Observe that the submonoid of (ℓ^k)th power maps

$$[\ell^{\mathbb{N}}] = \{[\ell^k] \ : \ k \in \mathbb{N}\} \subset \mathrm{End}_\ell(C)$$

acts freely on $\mathrm{Sub}_\ell(C)$. We define the set of *virtual subgroups of* $C[\ell^\infty]$ to be the the set

$$\mathrm{Sub}_\ell^0(C) = \mathrm{Sub}_\ell(C)[\ell^{-1}] = \{[\ell^k] \cdot H \ : \ k \in \mathbb{Z}, H \in \mathrm{Sub}_\ell(C)\}$$

where we have inverted the action of $[\ell]$. Because the group Γ is given by inverting ℓ in the monoid $\mathrm{End}_\ell(C)$ (Lemma 2.5.1), we see that the action of $\mathrm{End}_\ell(C)$ on $\mathrm{Sub}_\ell^0(C)$ extends to an action of the group Γ. Explicitly, for a quasi-isogeny $\psi = \ell^k \phi$ and a virtual subgroup $H = \ell^{k'} \widetilde{H}$, where $\phi \in \mathrm{End}_\ell(C)$ and $\widetilde{H} \in \mathrm{Sub}_\ell(C)$, this action is given by

$$\psi \cdot H = \widehat{\psi}^{-1}(H) := [\ell^{k+k'}] \cdot \widehat{\phi}^{-1}(\widetilde{H}).$$

As described in Section 2.7, the set $\mathrm{Sub}_\ell(C)$ possesses a natural Galois action through the action of Frob_p^{rel} on $C[\ell^\infty]$

$$\sigma \cdot H = \mathrm{Frob}_p^{rel}(H)$$

and this action extends to $\mathrm{Sub}_\ell^0(C)$. By Lemma 2.7.1, this action is compatible with the Galois action on Γ, giving $\mathrm{Sub}_\ell^0(C)$ the structure of a Γ_{Gal}-set.

The cardinality map

$$\mathrm{ord} : \mathrm{Sub}_\ell(C) \to \ell^{\mathbb{N}},$$

which takes a subgroup to its order, extends to a map

$$\mathrm{ord} : \mathrm{Sub}_\ell^0(C) \to \ell^{\mathbb{Z}}.$$

This map is Γ-equivariant, where $\phi \in \Gamma$ acts on the right-hand side by multiplication by the degree $N(\phi)$.

Let $\mathcal{L}(V_\ell(C))$ be the set of lattices in $V_\ell(C)$. It is a Γ-set under the inclusion $\Gamma \hookrightarrow GL(V_\ell(C))$, and the compatible Galois action (Section 2.7) induces a Γ_{Gal}-action on $\mathcal{L}(V_\ell(C))$.

Proposition 3.2.1. There is a Γ_{Gal}-equivariant isomorphism

$$\kappa : \mathcal{L}(V_\ell(C)) \to \mathrm{Sub}_\ell^0(C).$$

Proof. The map κ is the composite

$$\kappa : \mathcal{L}(V_\ell(C)) \xrightarrow{\widetilde{e}} \mathcal{L}(C[\ell^\infty]^* \otimes \mathbb{Q}) \xrightarrow{\mathrm{ker}} \mathrm{Sub}_\ell^0(C)$$

where $\widetilde{e}$ is the Galois equivariant isomorphism of Proposition 2.1.4, under which $C[\ell^\infty]^*$ inherits the Γ action given in Lemma 2.1.5. We describe the map ker. If we are given a lattice $L \subset C[\ell^\infty]^* \otimes \mathbb{Q}$ which is actually contained in $C[\ell^\infty]^*$, we define $\mathrm{ker}(L)$ to be the subgroup of C given by

$$\mathrm{ker}(L) = \bigcap_{\alpha \in L} \mathrm{ker}\, \alpha \in \mathrm{Sub}_\ell(C).$$

If L is a general lattice in $C[\ell^\infty]^* \otimes \mathbb{Q}$, then there exists a k such that $\ell^k L$ is contained in $C[\ell^\infty]^*$. Define $\mathrm{ker}(L)$ to be the virtual subgroup

$$\mathrm{ker}(L) = [\ell^{-k}] \cdot \mathrm{ker}(\ell^k L).$$

This is easily seen to be independent of the choice of k.

In order to show that ker is Γ-equivariant, it suffices to check the Γ-equivariance on elements ϕ of Γ contained in $\mathrm{End}_\ell(C)$ and on lattices contained in $C[\ell^\infty]^*$. We

then have

$$\ker(\phi \cdot L) = \ker(\{\alpha \circ \widehat{\phi} \ : \ \alpha \in L\})$$
$$= \bigcap_{\alpha \in L} \ker(\alpha \circ \widehat{\phi})$$
$$= \bigcap_{\alpha \in L} \widehat{\phi}^{-1}(\ker(\alpha))$$
$$= \widehat{\phi}^{-1}(\ker(L))$$
$$= \phi \cdot \ker(L).$$

The Galois equivariance of ker is easily verified:

$$\ker(\sigma \cdot L) = \bigcap_{\alpha \in L} \ker(\mathrm{Frob}_p^{rel} \circ \alpha \circ (\mathrm{Frob}_p^{rel})^{-1})$$
$$= \bigcap_{\alpha \in L} \ker(\alpha \circ (\mathrm{Frob}_p^{rel})^{-1})$$
$$= \bigcap_{\alpha \in L} \mathrm{Frob}_p^{rel}(\ker \alpha)$$
$$= \mathrm{Frob}_p^{rel}(\ker(L))$$
$$= \sigma \cdot \ker(L).$$

To see that the map ker is a bijection, we construct an inverse. Given a finite subgroup H of $C[\ell^\infty]$, define L_H to be the subgroup of $C[\ell^\infty]^*$ given by

$$L_H = \{\alpha \in C[\ell^\infty]^* \ : \ \alpha(H) = 1\}.$$

We claim that L_H is a lattice, that is, that the inclusion

$$L_H \otimes \mathbb{Q} \hookrightarrow C[\ell^\infty]^* \otimes \mathbb{Q}$$

is a bijection. It suffices to show that there exists a positive integer k such that $C[\ell^\infty]^*$ is contained in $\ell^{-k} L_H$, or equivalently, such that $\ell^k C[\ell^\infty]^*$ is contained in L_H. This is accomplished by choosing k such that H is contained in the ℓ^k torsion of C. The mapping $H \mapsto L_H$ extends to a map

$$\mathrm{Sub}_\ell^0(C) \to \mathcal{L}(C[\ell^\infty]^* \otimes \mathbb{Q})$$

which is inverse to the map ker. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Suppose that $H = [\ell^k] \cdot \widetilde{H}$ and $H' = [\ell^{k'}] \cdot \widetilde{H}'$ are virtual subgroups of $C[\ell^\infty]$, for $\widetilde{H}, \widetilde{H}' \in \mathrm{Sub}_\ell(C)$. We may assume that $k = k'$. We shall say that H is *contained* in H' and write

$$H \leq H'$$

if $\widetilde{H}$ is contained in $\widetilde{H}'$. Define the *quotient* to be

$$H'/H = \widetilde{H}'/\widetilde{H}.$$

Observe that this depends on the choice of k, but any two choices of k will give canonically isomorphic quotients.

Lemma 3.2.2. Suppose that $L_0 \leq L_1$ are lattices in $V_\ell(C)$. Then there is a containment

$$\kappa(L_1) \leq \kappa(L_0)$$

of virtual subgroups and a (non-canonical) isomorphism

$$L_1/L_0 \cong \kappa(L_0)/\kappa(L_1)$$

between the quotients.

Proof. Let the lattices $\widetilde{e}(L_i) \subset C[\ell^\infty]^* \otimes \mathbb{Q}$ be the images of the lattices L_i under the map $\widetilde{e}$ of Proposition 2.1.4. We may as well assume that the lattices $\widetilde{e}(L_i)$ are contained in $C[\ell^\infty]^*$. The subgroups $\kappa(L_i)$ are the kernels of the dual projections

$$0 \to \kappa(L_i) \to C[\ell^\infty] \to \widetilde{e}(L_i)^* \to 0.$$

There are therefore isomorphisms

$$(L_1/L_0)^* \cong \ker(L_1^* \twoheadrightarrow L_0^*) \cong \kappa(L_0)/\kappa(L_1)$$

using the exactness of the Pontryagin dual and the 3×3 lemma. Since L_1/L_0 is finite, it is non-canonically isomorphic to its Pontryagin dual $(L_1/L_0)^*$. $\qquad\square$

3.3. Construction of $\mathcal{J}'$ using virtual subgroups.

The map κ of Proposition 3.2.1 and Lemma 3.2.2 gives the following alternative description of the sets of simplices in the semi-simplicial set

$$\mathcal{J}_0' \Leftarrow \mathcal{J}_1' \Lleftarrow \mathcal{J}_2'$$

in terms of flags of virtual subgroups of $C[\ell^\infty]$:

$$\mathcal{J}_0' = \{H_0 \ : \ H_0 \text{ a virtual subgroup of } C[\ell^\infty]\},$$
$$\mathcal{J}_1' = \{H_1 < H_0 \ : \ H_0/H_1 \cong \mathbb{Z}/\ell \text{ or } \mathbb{Z}/\ell \times \mathbb{Z}/\ell\},$$
$$\mathcal{J}_2' = \{H_2 < H_1 < H_0 \ : \ H_1/H_2 \cong \mathbb{Z}/\ell \text{ and } H_0/H_2 \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell\}.$$

The ith face maps are given by deleting the ith terms of the flags. This semisimplicial set is Γ equivariant with the group acting by permuting the flags. This action agrees with the action given by the inclusion $\Gamma \hookrightarrow GL(V_\ell(C))$ since the map κ was proven to be Γ-equivariant.

3.4. The Γ orbits in $\mathcal{J}'$.

We shall explicitly identify the Γ orbits of the sets $\mathcal{J}_i'$, and determine their isotropy.

Recall that X^{ss} is the set of isomorphism classes of supersingular elliptic curves C' defined over $\overline{\mathbb{F}}_p$, and $X_0^{ss}(\ell)$ is the set of isomorphism classes of pairs (C', H) of supersingular curves C' with a cyclic subgroup H of order ℓ. Fix representatives of these isomorphism classes. We shall make use of the following result of Kohel.

Theorem 3.4.1 (Kohel, [20, Cor. 77]). *Let C' and C'' be supersingular elliptic curves over $\overline{\mathbb{F}}_p$. Then for all $k \gg 0$, there exists an isogeny $\phi : C' \to C''$ of degree ℓ^k.*

Since there are finitely many elliptic curves, there exists an $e > 0$ so that we may choose isogenies

$$\phi_{C'} : C \to C'$$

of degree ℓ^{2e} for every $C' \in X^{ss}$. We may as well assume that $\phi_C = [\ell^e]$. This uniformity in the degrees of the isogenies $\phi_{C'}$ has the effect of simplifying some of our proofs. Our insistence on using isogenies of degree an even power of ℓ will come into play in Section 4.3 (see Lemma 4.3.1).

Our choices of the isogenies $\phi_{C'}$ give embeddings

$$\iota_{C'} : \text{Aut}(C') \hookrightarrow \Gamma$$

of the automorphism groups $\text{Aut}(C')$ for $C' \in X^{ss}$. Given an automorphism $\gamma \in \text{Aut}(C')$, let $\iota_{C'}(\gamma)$ be the quasi-isogeny of C given by

$$\iota_{C'}(\gamma) = [\ell^{-2e}] \cdot (\widehat{\phi}_{C'} \circ \gamma \circ \phi_{C'}).$$

The factor of $[\ell^{-2e}]$ makes $\iota_{C'}$ a homomorphism of groups. For $(C', H) \in X_0^{ss}(\ell)$ we regard the subgroups $\text{Aut}(C', H)$ of $\text{Aut}(C')$ to be embedded in Γ by $\iota_{C'}$.

Proposition 3.4.2. The Γ-sets $\mathcal{J}_i'$ decompose into Γ-orbits as follows:

$$\mathcal{J}_0' = \coprod_{C' \in X^{ss}} \mathcal{J}_0'[C'],$$

$$\mathcal{J}_1' = \coprod_{(C',H) \in X_0^{ss}(\ell)} \mathcal{J}_1'[C', H] \quad \text{II} \quad \coprod_{C' \in X^{ss}} \mathcal{J}_1'[C'],$$

$$\mathcal{J}_2' = \coprod_{(C',H) \in X_0^{ss}(\ell)} \mathcal{J}_2'[C', H].$$

These orbits are given as follows:

$$\mathcal{J}_0'[C'] = \{H_0 \; : \; C/\widetilde{H}_0 \cong C'\},$$

$$\mathcal{J}_1'[C', H] = \{H_1 < H_0 \; : \; (C/\widetilde{H}_0, \ell \cdot \widetilde{H}_1/\widetilde{H}_0) \cong (C', H)\},$$

$$\mathcal{J}_1'[C'] = \{H_1 < H_0 \; : \; C/\widetilde{H}_0 \cong C' \text{ and } H_0 = \ell \cdot H_1\},$$

$$\mathcal{J}_2'[C', H] = \{H_2 < H_1 < H_0 \; : \; (C/\widetilde{H}_0, \ell \cdot \widetilde{H}_1/\widetilde{H}_0) \cong (C', H)$$
$$\text{and } H_0 = \ell \cdot H_2\}$$

where the subgroups $\widetilde{H}_i$ of C are obtained from the virtual subgroups H_i by multiplying by a suitable uniform power of ℓ.

Proof. We explain the decomposition of $\mathcal{J}_0'$. The arguments are essentially the same for the other $\mathcal{J}_i'$. We first verify that the set $\mathcal{J}_0'[C']$ is closed under the action of Γ. It suffices to check that if H_0 is a subgroup of C such that $C/H_0 \cong C'$ and ϕ is an isogeny in $\text{End}_\ell(C)$, then $\phi \cdot H_0 = \widehat{\phi}^{-1}(H_0)$ has the property that there is an isomorphism $C/\widehat{\phi}^{-1}(H_0) \cong C'$. This isomorphism exists because $\widehat{\phi}^{-1}(H_0)$ is the kernel of the composite

$$C \xrightarrow{\widehat{\phi}} C \to C/H_0 \xrightarrow{\cong} C'.$$

We now verify transitivity. Suppose that H_0 and H_0' are elements of $\mathcal{J}_0'[C']$. We may assume that the virtual subgroups H_i are actually subgroups. Let ϕ_{H_0} and $\phi_{H_0'}$ be the quotient maps

$$\phi_{H_0} : C \to C/H_0,$$

$$\phi_{H_0'} : C \to C/H_0'.$$

By hypothesis, there is an isomorphism

$$\gamma : C/H_0 \xrightarrow{\cong} C/H_0'.$$

Let ψ be the composite

$$\psi : C \xrightarrow{\phi_{H_0}} C/H_0 \xrightarrow{\gamma} C/H_0' \xrightarrow{\widehat{\phi}_{H_0'}} C.$$

Then we have

$$\psi \cdot H_0 = [\ell^i] \cdot H_0'$$

where ℓ^i is the order of H_0, so $(\ell^{-i}\psi) \cdot H_0 = H_0'$. The action of Γ is therefore transitive. $\qquad\square$

Proposition 3.4.3. The orbits of Proposition 3.4.2 are identified as follows:

$$\mathcal{J}_i'[C'] \cong \Gamma/\operatorname{Aut}(C'),$$
$$\mathcal{J}_i'[C', H] \cong \Gamma/\operatorname{Aut}(C', H).$$

Proof. For each C' in X^{ss}, let $K_{C'}$ be the kernel of the isogeny $\phi_{C'}$. For each (C', H) in $X_0^{ss}(\ell)$, let $K_{(C',H)}$ be the kernel of the composite

$$C \xrightarrow{\phi_{C'}} C' \twoheadrightarrow C'/H.$$

Then the stabilizers of certain well-chosen points of $\mathcal{J}_i'$ are easily determined:

$$\operatorname{Stab}_\Gamma(K_{C'}) = \operatorname{Aut}(C'),$$
$$\operatorname{Stab}_\Gamma(\ell^{-1} \cdot K_{(C',H')} < K_{C'}) = \operatorname{Aut}(C', H),$$
$$\operatorname{Stab}_\Gamma(\ell^{-1} \cdot K_{C'} < K_{C'}) = \operatorname{Aut}(C'),$$
$$\operatorname{Stab}_\Gamma(\ell^{-1} \cdot K_{C'} < \ell^{-1} \cdot K_{(C',H)} < K_{C'}) = \operatorname{Aut}(C', H).$$

$\qquad\square$

Combining Propositions 3.4.2 and 3.4.3, we have the following theorem.

Theorem 3.4.4. There are Γ_{Gal}-equivariant isomorphisms

$$\mathcal{J}_0' \cong \coprod_{C' \in X^{ss}} \Gamma/\operatorname{Aut}(C'),$$

(3.4.5)
$$\mathcal{J}_1' \cong \coprod_{(C',H) \in X_0^{ss}(\ell)} \Gamma/\operatorname{Aut}(C', H) \quad \amalg \coprod_{C' \in X^{ss}} \Gamma/\operatorname{Aut}(C'),$$

$$\mathcal{J}_2' \cong \coprod_{(C',H) \in X_0^{ss}(\ell)} \Gamma/\operatorname{Aut}(C', H).$$

Proof. We are only left with verifying the Galois equivariance of these isomorphisms. We will only treat the case of $\mathcal{J}_i'$ with $i = 0$. The cases of $i > 0$ are completely analogous.

Recall that we have embedded $\operatorname{Aut}(C')$ in Γ by conjugating with $\phi_{C'}$. Our practice of denoting the Γ orbit corresponding to C' by $\Gamma/\operatorname{Aut}(C')$ is misleading, because it conceals the manner in which we have embedded $\operatorname{Aut}(C')$. The orbit is more precisely given by

$$\Gamma/\phi_{C'}^{-1}\operatorname{Aut}(C')\phi_{C'}.$$

The decomposition of $\mathcal{J}_0'$ is given by the composite

$$\Gamma/\phi_{C'}^{-1}\operatorname{Aut}(C')\phi_{C'} \xrightarrow[f]{\cong} \Gamma \cdot K_{C'} \xrightarrow{\cong} \mathcal{J}_0'[C']$$

where $K_{C'} \in \mathrm{Sub}_\ell^0(C)$ is the kernel of $\phi_{C'}$. The map f is given by

$$f(x\phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'}) = x \cdot K_{C'}$$

for $x \in \Gamma$.

The action of $\sigma \in Gal$

$$\sigma : \mathcal{J}'[C'] \to \mathcal{J}'[\sigma_* C']$$

is given by

$$\sigma \cdot H = \mathrm{Frob}_p^{rel}(H)$$

for $H \in \mathrm{Sub}_\ell(C)$ with $C/H \cong C'$. Using Lemma 2.7.1, we have

$$\mathrm{Frob}_p^{rel}(K_{C'}) = y_{C'} \cdot K_{\sigma_* C'}$$

for

$$y_{C'} = \ell^{-2e} \cdot (\sigma_* \widehat{\phi}_{C'}) \cdot \phi_{\sigma_* C'} \in \Gamma.$$

Thus the compatible σ action

$$\sigma : \Gamma \cdot K_{C'} \to \Gamma \cdot K_{\sigma_* C'}$$

is given by

$$\begin{aligned}
\sigma \cdot (x \cdot K_{C'}) &= (\sigma_* x) \cdot \sigma \cdot K_{C'} \\
&= (\sigma_* x) \cdot y_{C'} \cdot K_{\sigma_* C'}.
\end{aligned}$$

We now compute the image of the subgroup $\phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'}$ under the action of σ_* on Γ.

$$\begin{aligned}
\sigma_*(\phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'}) &= \mathrm{Frob}_p \, \phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'} \, \mathrm{Frob}_p^{-1} \\
&= (\sigma_* \phi_{C'})^{-1} \mathrm{Frob}_p \, \mathrm{Aut}(C') \, \mathrm{Frob}_p^{-1}(\sigma_* \phi_{C'}) \\
&= (\sigma_* \phi_{C'})^{-1} \mathrm{Aut}(\sigma_* C')(\sigma_* \phi_{C'}) \\
&= y_{C'} \phi_{\sigma_* C'}^{-1} \mathrm{Aut}(\sigma_* C')\phi_{\sigma_* C'} y_{C'}^{-1}.
\end{aligned}$$

With this in mind, the natural action of σ_* on $\Gamma/\phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'}$ is given by

(3.4.6) $$\begin{aligned}
\sigma : \Gamma/\phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'} &\to \Gamma/\phi_{\sigma_* C'}^{-1} \mathrm{Aut}(\sigma_* C')\phi_{\sigma_* C'} \\
x \cdot \phi_{C'}^{-1} \mathrm{Aut}(C')\phi_{C'} &\mapsto \sigma_* x \cdot y_{C'} \cdot \phi_{\sigma_* C'}^{-1} \mathrm{Aut}(\sigma_* C')\phi_{\sigma_* C'}.
\end{aligned}$$

This action makes the map f Galois equivariant. $\qquad\square$

3.5. The semi-simplicial structure of $\mathcal{J}'$.

In Section 3.4 we gave a Γ-equivariant orbit decomposition of the n-simplices of $\mathcal{J}'_\bullet$. In this section we shall describe the face maps in terms of this orbit decomposition.

We shall first need some definitions. Recall from Section 3.4 that we have fixed representatives C' (respectively (C', H)) for each isomorphism class of X^{ss} (respectively $X_0^{ss}(\ell)$). We also fixed isogenies

$$\phi_{C'} : C \to C'$$

of degree ℓ^{2e} for each $C' \in X^{ss}$.

For each pair $(C', H) \in X_0^{ss}(\ell)$, there is an induced degree ℓ isogeny given by the quotient

$$q_H : C' \to C'/H.$$

Let $\widehat{H} \subset C'/H$ be the kernel of the dual isogeny $\widehat{q}_H$. Then there exists a pair $(C_H, d(H)) \in X_0^{ss}(\ell)$ and an isomorphism

$$\alpha_H : C'/H \to C_H$$

which sends $\widehat{H}$ to $d(H)$. Define ϕ_H to be the composite

$$\phi_H : C' \xrightarrow{q_H} C'/H \xrightarrow{\alpha_H} C_H.$$

Then $d(H)$ is the kernel of the dual isogeny $\widehat{\phi}_H$.

For each pair $(C', H) \in X_0^{ss}(\ell)$, define elements of $g_{(C',H)}$ of Γ by

(3.5.1) $$g_{(C',H)} = \ell^{-2e-1} \cdot (\widehat{\phi}_{C'} \circ \widehat{\phi}_H \circ \phi_{C_H}).$$

Proposition 3.5.2. Under the isomorphisms of Equation (3.4.5), the face maps of the semisimplicial set $\mathcal{J}'_\bullet$ are given as follows.

$$\begin{aligned}
d_i : &\mathcal{J}'_1 \to \mathcal{J}'_0 \\
&d_0(x \operatorname{Aut}(C', H)) = x g_{(C',H)} \operatorname{Aut}(C_H), \\
&d_1(x \operatorname{Aut}(C', H)) = x \operatorname{Aut}(C'), \\
&d_0(x \operatorname{Aut}(C')) = x \cdot \ell^{-1} \operatorname{Aut}(C'), \\
&d_1(x \operatorname{Aut}(C')) = x \operatorname{Aut}(C'). \\
d_i : &\mathcal{J}'_2 \to \mathcal{J}'_1 \\
&d_0(x \operatorname{Aut}(C', H)) = x g_{(C',H)} \operatorname{Aut}(C_H, d(H)), \\
&d_1(x \operatorname{Aut}(C', H)) = x \operatorname{Aut}(C'), \\
&d_2(x \operatorname{Aut}(C', H)) = x \operatorname{Aut}(C', H).
\end{aligned}$$

Proof. We simply must evaluate the face maps of $\mathcal{J}'$, as given in Section 3.3 on the orbit representatives chosen in the proof of Proposition 3.4.3:

$$\begin{aligned}
d_0(\ell^{-1} K_{(C',H)} < K_{C'}) &= \ell^{-1} K_{(C',H)} \\
&= g_{(C',H)} \cdot K_{C_H}, \\
d_1(\ell^{-1} K_{(C',H)} < K_{C'}) &= K_{C'}, \\
d_0(\ell^{-1} K_{C'} < K_{C'}) &= \ell^{-1} K_{C'}, \\
d_1(\ell^{-1} K_{C'} < K_{C'}) &= K_{C'}, \\
d_0(\ell^{-1} K_{C'} < \ell^{-1} K_{(C',H)} < K_{C'}) &= \ell^{-1} K_{C'} < \ell^{-1} K_{(C',H)} \\
&= g_{(C',H)} \cdot (\ell^{-1} K_{(C_H, d(H))} < K_{C_H}), \\
d_1(\ell^{-1} K_{C'} < \ell^{-1} K_{(C',H)} < K_{C'}) &= \ell^{-1} K_{C'} < K_{C'}, \\
d_2(\ell^{-1} K_{C'} < \ell^{-1} K_{(C',H)} < K_{C'}) &= \ell^{-1} K_{(C',H)} < K_{C'}.
\end{aligned}$$

$\square$

4. THE BUILDING FOR $SL_2(\mathbb{Q}_\ell)$

In this section we recall the construction of the building $\mathcal{J}$ for $SL_2(\mathbb{Q}_\ell)$. We then give a reinterpretation in terms of virtual subgroups of $C[\ell^\infty]$ which is more amenable to understanding the action of the subgroup Γ^1 of $SL(V_\ell(C))$.

We decompose $\mathcal{J}$ into orbits under the action of Γ^1, and demonstrate that this group acts on $\mathcal{J}$ with finite stabilizers. We then explain how Bass-Serre theory gives the structure of the group Γ^1 as the fundamental groups of a graph of finite groups. Much of the material in this section may be found in Serre's book [32].

4.1. The construction of $\mathcal{J}$ using lattices.

Let V be a $\mathbb{Q}_\ell$ vector space of dimension 2. Two lattices L and L' in V are said to be *homothetic* if there exists a $c \in \mathbb{Q}_\ell^\times$ such that

$$L' = cL.$$

Since L and L' are $\mathbb{Z}_\ell$-modules, c may be chosen to be ℓ^k for some integer k. The construction of $\mathcal{J}$ follows the construction of $\mathcal{J}'$ except that we use homothety classes of lattices in V. The building $\mathcal{J}$ is a 1-dimensional contractible simplicial complex on which $SL(V)$ acts. Topologically, $\mathcal{J}$ is an $\ell + 1$-regular tree.

Specifically, $\mathcal{J}$ is the geometric realization of a semi-simplicial $SL(V)$ set of the form

$$\mathcal{J}_0 \Leftarrow \mathcal{J}_1$$

where the sets $\mathcal{J}_i$ are sets of flags of homothety classes of lattices in V:

$\mathcal{J}_0 = \{[L_0] \ : \ [L_0] \text{ a homothety class of lattice in } V\}$,

$\mathcal{J}_1 = \{\{[L_0], [L_1]\} \ : \ \text{there exist reps } L_0 < L_1 \text{ such that } L_1/L_0 \cong \mathbb{Z}/\ell\}$.

The group $GL(V)$ acts by permuting the lattice classes in the flags. This action restricts to an action of $SL(V)$. Since we are taking homothety classes of lattices, the center $\mathbb{Q}_\ell^\times \subseteq GL(V)$ acts trivially on $\mathcal{J}$, so the action also factors through $PGL(V)$.

There is a $GL(V)$-equivariant projection

$$\nu : \mathcal{J}' \to \mathcal{J}$$

given by taking homothety classes of the lattices that make up the flags of $\mathcal{J}'$. Under this map, the simplices of $\mathcal{J}_1'$ corresponding to flags $L_0 < L_1$ with $L_1/L_0 \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell$, as well as all of the simplices of $\mathcal{J}_2'$, become degenerate.

4.2. The construction of $\mathcal{J}$ using virtual subgroups.

Let $V = V_\ell(C)$. The same methods that construct $\mathcal{J}'$ in terms of virtual subgroups construct $\mathcal{J}$ in terms of homothety classes of virtual subgroups of $C[\ell^\infty]$. Here, two virtual subgroups H and H' are said to be homothetic if there exists an integer k such that

$$H' = [\ell^k] \cdot H.$$

Lemma 4.2.1. Every virtual subgroup H of $C[\ell^\infty]$ is homothetic to a unique virtual subgroup H' where the order of H' is either 1 or ℓ. The virtual subgroup H' is uniquely expressible in the form

$$H' = [\ell^k] \cdot H''$$

for some integer k, where H'' is a subgroup of C isomorphic to $\mathbb{Z}/\ell^m$.

Proof. The virtual subgroup H' is $[\ell^{-i}] \cdot H$ where the order of H is either ℓ^{2i} or ℓ^{2i+1}. To produce the canonical representative H'', we may as well assume that the representative H of the homothety class $[H]$ is a subgroup. Let j be maximal so that the ℓ^j-torsion subgroup $C[\ell^j]$ is contained in H. Then the subgroup H'' is given by

$$H'' = H/C[\ell^j] \subset C/C[\ell^j].$$

We may regard H'' as being contained in C under the canonical isomorphism $[\ell^j]$: $C/C[\ell^j] \xrightarrow{\cong} C$. $\square$

If the order of the group H' given by Lemma 4.2.1 is 1, we shall say the homothety class $[H]$ is *even*. Otherwise we shall say that the homothety class $[H]$ is *odd*.

Observe that the isomorphism κ of Proposition 3.2.1 identifies homothety classes of lattices with homothety classes of virtual subgroups. The semi-simplicial set

$$\mathcal{J}_0 \Leftarrow \mathcal{J}_1$$

whose realization is $\mathcal{J}$ may therefore be described in terms of virtual subgroups:

$$\mathcal{J}_0 = \{[H_0] \ : \ [H_0] \text{ a homothety class of virtual subgroup in } C[\ell^\infty]\},$$
$$\mathcal{J}_1 = \{([H_1],[H_0]) \ : \ \text{there exist reps } H_1 < H_0 \text{ such that } H_0/H_1 \cong \mathbb{Z}/\ell,$$
$$[H_0] \text{ even}, [H_1] \text{ odd}\}.$$

The group Γ^1 acts by permuting the classes of virtual subgroups.

4.3. **The Γ^1 orbit decomposition of $\mathcal{J}$.** We must first remark that Γ^1 contains only half of the isogenies of Γ modulo $[\ell^{\mathbb{Z}}]$.

Lemma 4.3.1. Every quasi-isogeny $\phi \in \Gamma^1$ is expressible uniquely in the form

$$\phi = \ell^{-i}\phi'$$

where ϕ' is an endomorphism of C whose kernel is isomorphic to $\mathbb{Z}/\ell^{2i}$.

Proof. There is a short exact sequence

$$1 \to \Gamma^1 \to \Gamma \xrightarrow{N} \ell^{\mathbb{Z}} \to 1.$$

The lemma is immediate from the fact that $\Gamma = \text{End}_\ell(C)[\ell^{-1}]$ and $N([\ell]) = \ell^2$. $\square$

The orbits of $\mathcal{J}_i$ are given in the following proposition, whose proof is completely analogous to that of Proposition 3.4.2. The decomposition of $\mathcal{J}_0$ into the two parity classes of Γ^1 orbits is a consequence of Lemmas 4.2.1 and 4.3.1.

Proposition 4.3.2. The Γ^1-sets $\mathcal{J}_i$ decompose into Γ^1-orbits as follows:

$$\mathcal{J}_0 = \coprod_{C' \in X^{ss}} (\mathcal{J}_0[C']_{even} \amalg \mathcal{J}_0[C']_{odd}),$$
$$\mathcal{J}_1 = \coprod_{(C',H) \in X_0^{ss}(\ell)} \mathcal{J}_1[C',H].$$

These orbits are given as follows:

$\mathcal{J}_0[C']_{even} = \{[H_0] \ : \ C/H_0 \cong C' \text{ and } [H_0] \text{ even}\},$
$\mathcal{J}_0[C']_{odd} = \{[H_0] \ : \ C/H_0 \cong C' \text{ and } [H_0] \text{ odd}\},$
$\mathcal{J}_1[C',H] = \{([H_1],[H_0]) \ : \ \text{there exist reps } H_1 < H_0 \text{ such that } H_0/H_1 \cong \mathbb{Z}/\ell,$
$$[H_0] \text{ even}, [H_1] \text{ odd}, (C/H_0, \ell \cdot H_1/H_0) \cong (C',H)\}.$$

Recall from Section 3.4 that we have embedded the group $\text{Aut}(C')$ as a subgroup of Γ^1 by conjugating by the isogeny $\phi_{C'}$:

$$\iota_{C'} : \text{Aut}(C') \hookrightarrow \Gamma^1$$
$$\alpha \mapsto \phi_{C'}^{-1}\alpha\phi_{C'}.$$

Fix an endomorphism ϕ of C of degree ℓ^{2r+1} for $r \gg 0$. Such an endomorphism exists by Theorem 3.4.1. We shall use $\overline{\mathrm{Aut}(C')}$ to denote the image of the different embedding of $\mathrm{Aut}(C')$ in Γ^1 given by

$$\bar{\iota}_{C'} : \overline{\mathrm{Aut}(C')} \hookrightarrow \Gamma^1$$

$$\alpha \mapsto \phi^{-1}\phi_{C'}^{-1}\alpha\phi_{C'}\phi.$$

The isotropy of $\mathcal{J}$ is described in the following proposition.

Proposition 4.3.3. The orbits of Proposition 4.3.2 are given by:

$$\mathcal{J}_0[C']_{even} \cong \Gamma^1/\mathrm{Aut}(C'),$$

$$\mathcal{J}_0[C']_{odd} \cong \Gamma^1/\overline{\mathrm{Aut}(C')},$$

$$\mathcal{J}_1[C', H] \cong \Gamma^1/\mathrm{Aut}(C', H).$$

Proof. For each C' in X^{ss}, let $K_{C'}^{even} = K_{C'}$ be the kernel of the isogeny $\phi_{C'}$, and let $K_{C'}^{odd}$ be the kernel of the composite

$$C \xrightarrow{\phi} C \xrightarrow{\phi_{C'}} C'.$$

For each (C', H) in $X_0^{ss}(\ell)$, let $K_{(C',H)}$ be the kernel of the composite

$$C \xrightarrow{\phi_{C'}} C' \twoheadrightarrow C'/H.$$

Then the stabilizers of certain well-chosen points of $\mathcal{J}_0[C']_{even}$, $\mathcal{J}_0[C']_{odd}$, and $\mathcal{J}_1[C', H]$ are easily determined:

$$\mathrm{Stab}_{\Gamma^1}([K_{C'}^{even}]) = \mathrm{Aut}(C'),$$

$$\mathrm{Stab}_{\Gamma^1}([K_{C'}^{odd}]) = \overline{\mathrm{Aut}(C')},$$

$$\mathrm{Stab}_{\Gamma^1}([K_{(C',H)}], [K_{C'}^{even}]) = \mathrm{Aut}(C', H).$$

$\square$

4.4. The semi-simplicial structure of $\mathcal{J}$. In this section we shall describe the face maps in the semi-simplicial set $\mathcal{J}_\bullet$ in terms of the orbit decomposition given in Section 4.3.

Combining Propositions 4.3.2 and 4.3.3, we have Γ^1-equivariant isomorphisms

$$\mathcal{J}_0 \cong \coprod_{C' \in X^{ss}} \Gamma^1/\mathrm{Aut}(C') \amalg \Gamma^1/\overline{\mathrm{Aut}(C')},$$

(4.4.1)

$$\mathcal{J}_1 \cong \coprod_{(C',H) \in X_0^{ss}(\ell)} \Gamma^1/\mathrm{Aut}(C', H).$$

For each pair $(C', H) \in X_0^{ss}(\ell)$, define an element $g_{(C',H)}^1$ of Γ^1 by

$$g_{(C',H)}^1 = \ell^{-(r+2e+1)} \cdot (\widehat{\phi}_{C'} \circ \widehat{\phi}_H \circ \phi_{C_H} \circ \phi).$$

Proposition 4.4.2. Under the isomorphisms of Equation (4.4.1), the face maps of the semisimplicial set $\mathcal{J}_\bullet$ are given as follows:

$$d_i : \mathcal{J}_1 \to \mathcal{J}_0$$

$$d_0(x\,\mathrm{Aut}(C', H)) = xg_{(C',H)}^1\overline{\mathrm{Aut}(C_H)},$$

$$d_1(x\,\mathrm{Aut}(C', H)) = x\,\mathrm{Aut}(C').$$

Proof. We evaluate the face maps of $\mathcal{J}$, as given in Section 4.2 on the orbit representatives chosen in the proof of Proposition 4.3.3:

$$d_0([K_{(C',H)}], [K_{C'}^{even}]) = [K_{(C',H)}]$$
$$= g_{(C',H)}^1 \cdot [K_{C_H}^{odd}],$$
$$d_1([K_{(C',H)}], [K_{C'}^{even}]) = [K_{C'}^{even}].$$

$\square$

4.5. The structure of Γ^1.

A graph of groups is a graph Y whose vertices and edges are labeled with finite groups, with inclusions compatible with the gluing data of the graph. In [32], the notion of the fundamental group of a connected graph of groups is given. If the graph Y is a tree, then this fundamental group is simply a suitable amalgamation of the labeling groups. We shall give a presentation of Γ^1 as the fundamental group of a graph of groups.

Let Y be the graph given by a semisimplicial set of the form

$$X^{ss} \amalg \overline{X}^{ss} \Leftarrow X_0^{ss}(\ell).$$

(Here, $\overline{X}^{ss}$ is X^{ss} — we have placed a bar over it to distinguish the two identical factors in the coproduct.) The face maps d_i are given by

$$d_0 : X_0^{ss}(\ell) \xrightarrow{t} \overline{X}^{ss} \hookrightarrow X^{ss} \amalg \overline{X}^{ss},$$
$$d_1 : X_0^{ss}(\ell) \xrightarrow{s} X^{ss} \hookrightarrow X^{ss} \amalg \overline{X}^{ss}$$

where the maps s and t are given on isomorphism classes by

$$s : [C', H] \mapsto [C'],$$
$$t : [C', H] \mapsto [C'/H].$$

We give Y the structure of a graph of groups $(Y, G_{(-)})$ by labeling the edges and vertices with groups as follows.

$$G_{[C']} = \mathrm{Aut}(C') \quad \text{for } [C'] \text{ in } X^{ss} \text{ or } \overline{X}^{ss}.$$
$$G_{[C',H]} = \mathrm{Aut}(C', H) \quad \text{for } [C', H] \text{ in } X_0^{ss}(\ell).$$

We associate to the face maps d_i of Y monomorphisms

$$(d_i)_* : G_{[C',H]} \to G_{d_i([C'])}.$$

The monomorphism d_1 is given by the natural inclusion

$$(d_1)_* \, \mathrm{Aut}(C', H) \hookrightarrow \mathrm{Aut}(C').$$

Any automorphism of C' which preserves a subgroup H descends to an automorphism of C'/H, and this gives the second of the two maps

$$(d_0)_* : \mathrm{Aut}(C', H) \to \mathrm{Aut}(C'/H).$$

Lemma 4.5.1. The map $(d_0)_*$ is a monomorphism.

Proof. Suppose that $(d_0)_*(\alpha) = \gamma = (d_0)_*(\alpha')$ for α and α' in $\mathrm{Aut}(C', H)$. Let ϕ be the quotient isogeny $C' \to C'/H$. The automorphism γ of C' satisfies

$$\phi \circ \alpha = \gamma \circ \phi = \phi \circ \alpha'.$$

By composing the above equation with the dual isogeny $\widehat{\phi}$, we see that there is an equality

$$\ell \cdot \alpha = \ell \cdot \alpha'$$

in the endomorphism ring $\mathrm{End}(C')$. Since this ring is torsion-free, we conclude that $\alpha = \alpha'$. $\qquad\square$

The group Γ^1 acts on the tree $\mathcal{J}$ without inversions. Proposition 4.3.2 shows that Y is the quotient $\Gamma^1 \backslash \mathcal{J}$. Bass-Serre theory [32, I.5.4], combined with Proposition 4.3.3 immediately gives the following theorem.

Theorem 4.5.2. The group Γ^1 is the fundamental group of the graph of groups $(Y, G_{(-)})$.

5. $K(2)$-LOCAL TOPOLOGICAL MODULAR FORMS

5.1. **Morava E-theories.** Goerss and Hopkins [16] refined the Hopkins-Miller Theorem [30] to produce a functor

$$E : \mathcal{FGL} \to E_\infty\text{-}ring\ spectra$$
$$(k, F) \mapsto E(k, F).$$

Here, $\mathcal{FGL}$ is the category of pairs (k, F), where k is a perfect field of characteristic p and F is a formal group of finite height over k. The spectrum $E(k, F)$ is complex orientable, and its associated formal group is the Lubin-Tate universal deformation F.

The Goerss-Hopkins-Miller functor extends naturally to the category of pairs (k, F) obtained by insisting that the ground ring $k = \prod_i k_i$ is only a product of perfect fields of characteristic p, via the assignment

$$E(k, F) = \prod_i E(k_i, F|_{k_i}).$$

In this paper, we are using E_n to denote the spectrum $E(\overline{\mathbb{F}}_p, H_n)$, where H_n is the Honda height n formal group. Functoriality gives rise to an action of the extended Morava stabilizer group

$$\mathbb{G}_n = \mathrm{Aut}(\overline{\mathbb{F}}_{p^n}, H_n) = \mathrm{Aut}(H_n) \rtimes Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p).$$

We remark that the subgroup $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$ of $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts trivially on $\mathrm{Aut}(H_n)$.

Our reason for working over $\overline{\mathbb{F}}_p$ is that formal groups over a separably closed field of positive characteristic p are classified by their height [21]. Therefore, given F, a formal group of height n over $\overline{\mathbb{F}}_p$, there is an isomorphism $\alpha : F \cong H_n$, and hence an isomorphism of E_∞-ring spectra $E(\overline{\mathbb{F}}_p, F) \simeq E(\overline{\mathbb{F}}_p, H_n) = E_n$, which depends on the isomorphism α.

5.2. **Homotopy fixed points.** Because we make extensive use of homotopy fixed point constructions, we pause to explain their meaning in the context of this paper. Let k be a finite extension of $\mathbb{F}_{p^n}$. Devinatz and Hopkins [12] gave a construction of homotopy fixed point spectra (which we shall denote $E_k^{\widetilde{h}H}$) of the spectrum $E_k = E(k, H_n)$ with respect to closed subgroups H of the profinite group

$$G_k = \mathrm{Aut}(k, H_n) = \mathrm{Aut}(H_n) \rtimes Gal(k/\mathbb{F}_p).$$

Actually, [12] is written in the context of $k = \mathbb{F}_{p^n}$, but there was nothing in the theory of [12] that prevented these authors from replacing $\mathbb{F}_{p^n}$ with the finite extension k.

Goerss and Hopkins [16] proved that for extensions k, k', the space of E_∞-ring maps $E_\infty(E_k, E_{k'})$ has contractible components. Thus the rectification methods of

Dwyer, Kan, and Smith [13],[12, 3.2] may be used to show that the construction of the spectrum $E_k^{\widetilde{h}H}$ may be made functorial in k.

For a profinite group G there is a more conventional notion of a *discrete G-spectrum* that has been investigated by Thomason, Jardine, Goerss, Davis and others (see, for instance, [9]). Let Set_G be the Grothendieck site of finite discrete G-sets. A discrete G-spectrum may be modeled as a presheaf of spectra on this site. The homotopy fixed points are given by Quillen derived functors of the global sections functor with respect to the model structure of [19]. Given a closed subgroup H of G, there is a restriction functor Res_H^G that takes presheaves of spectra on Set_G to presheaves of spectra on Set_H.

Following Daniel Davis [9], we shall regard the Devinatz-Hopkins construction as producing a presheaf $\mathcal{E}_n(-)$ of spectra on the site $\mathrm{Set}_{\mathbb{G}_n}$. For an open subgroup U of $\mathbb{G}_n$, let $W(U)$ be the subgroup

$$W(U) = U \cap Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n}) \leq Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n}) \lhd \mathbb{G}_n.$$

Define $k(U)$ to be the finite fixed field

$$k(U) = \overline{\mathbb{F}}_p^{W(U)}.$$

The value of the presheaf $\mathcal{E}(-)$ on the transitive finite discrete $\mathbb{G}_n$-set $\mathbb{G}_n/U$ is given by

$$\mathcal{E}_n(\mathbb{G}_n/U) = E_{k(U)}^{\widetilde{h}U/W(U)}$$

For H a closed subgroup of $\mathbb{G}_n$, we *define* the homotopy fixed point spectrum as the $K(n)$-localization of the derived global sections of the restricted presheaf

(5.2.1) $$E_n^{hH} = (R\Gamma \, \mathrm{Res}_H^{\mathbb{G}_n} \, \mathcal{E}_n)_{K(n)}.$$

Davis showed that these constructions are equivalent to those of Devinatz and Hopkins. The statement of his theorem given below is a mild extension of the statement which appears in [8].

Theorem 5.2.2 (Davis [8]). There is an equivalence $E_n^{\widetilde{h}H} \simeq E_n^{hH}$.

The Galois descent properties of E_n are axiomatized by Rognes [31]. In his language, the spectrum E_n is a $K(n)$-local profinite Galois extension of $S_{K(n)}$. The homotopy fixed points of such spectra are remarkably well behaved, as demonstrated in [3]. In particular we show that, when homotopy fixed point spectra are defined in the sense of Equation 5.2.1, we may iterate the homotopy fixed point construction.

Proposition 5.2.3 (Behrens-Davis [3]). For K a closed normal subgroup of H, a closed subgroup of $\mathbb{G}_n$, there is an equivalence $(E_n^{hK})^{hH/K} \simeq E_n^{hH}$.

Remark 5.2.4. Devinatz has investigated a different approach to iterated homotopy fixed points that differs philosophically from ours [11]. Namely, he *defines* the iterated fixed point construction $(E_n^{\widetilde{h}K})^{hH/K}$ to be the spectrum $E_n^{\widetilde{h}H}$ and then shows that this definition makes sense (e.g. there is an associated Lyndon-Hochschild-Serre spectral sequence).

Our reasons for engaging in this rhetorical yoga surrounding the construction of homotopy fixed points is twofold. Firstly, for Λ a discrete group which lies as a

subgroup in a profinite subgroup G, and for E a discrete G-spectrum, there is a natural map

$$E^{hG} \to E^{h\Lambda}$$

where the spectrum $E^{h\Lambda}$ is the ordinary homotopy fixed point spectrum. Producing this map using only the Devinatz-Hopkins language is less transparent. Secondly, in this language, we can employ the following lemma more freely.

Theorem 5.2.5 (Goerss [14, Theorem 6.1])**.** Suppose that E is a discrete $\widehat{\mathbb{Z}}$-spectrum. Then the natural map

$$E^{h\widehat{\mathbb{Z}}} \to E^{h\mathbb{Z}}$$

is an $H\mathbb{F}_p$-equivalence.

Goerss actually proved this theorem in the context of spaces, but the case of spectra is handled by similar means, and is in some sense easier.

5.3. **Topological modular forms: an overview.** Let $\mathcal{M}$ be the moduli stack of generalized elliptic curves. Goerss, Hopkins, Miller and their collaborators have constructed a sheaf $\mathcal{O}_{ell}$ (in the étale topology) of E_∞-ring spectra over $\mathcal{M}$. The spectrum tmf is given by the connective cover of the global sections

$$tmf = \tau_{\geq 0} \mathcal{O}_{ell}(\mathcal{M}).$$

The global sections $\mathcal{O}_{ell}(\mathcal{M})$ then give the $E(2)$-localization $tmf_{E(2)}$. Let $\mathcal{M}^{ns}$ be the substack of non-singular elliptic curves. The spectrum TMF is the spectrum of sections $\mathcal{O}_{ell}(\mathcal{M}^{ns})$. Let $\mathcal{M}^{ss}$ be a formal neighborhood of the mod p supersingular locus of $\mathcal{M}$. Then the $K(2)$-localization $tmf_{K(2)} = TMF_{K(2)}$ is the spectrum of sections

$$TMF_{K(2)} = \mathcal{O}_{ell}(\mathcal{M}^{ss}).$$

Let $\mathcal{M}_0(\ell)$ be the moduli stack of elliptic curves with $\Gamma_0(\ell)$-structures. The forgetful map

$$\phi_f : \mathcal{M}_0^{ss}(\ell) \to \mathcal{M}^{ss}$$

is étale, so we may evaluate $\mathcal{O}_{ell}$ on ϕ_f to realize $TMF_0(\ell)_{K(2)}$ as the spectrum of sections $\mathcal{O}_{ell}(\mathcal{M}_0^{ss}(\ell))$.

Because a detailed account of this story does not yet exist in the literature, we reproduce just enough of it to give the constructions, due to Goerss, Hopkins, Miller, and their collaborators, of $TMF_{K(2)}$ and $TMF_0(\ell)_{K(2)}$ that we require. What follows is basically a recapitulation of a lecture of Charles Rezk on the subject in a workshop on topological modular forms held in Münster, Germany in 2003.

5.4. **The neighborhood of the supersingular locus.** Let $\mathbb{W} = \mathbb{W}(\overline{\mathbb{F}}_p)$ be the Witt ring with residue field $\overline{\mathbb{F}}_p$. We shall first describe the stack $\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$. This formal stack is a profinite Galois covering of the stack $\mathcal{M}^{ss}$, with covering group equal to $Gal = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Thus we may recover the sections of sheaves (in the étale topology) over $\mathcal{M}^{ss}$ from their sections over $\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$ by taking Galois invariants.

For each isomorphism class $[C']$ in X^{ss} we choose a representative C' defined over $\overline{\mathbb{F}}_p$. Let k be the perfect ring given by the product $\prod_{X^{ss}} \overline{\mathbb{F}}_p$. Let $\mathbf{C}$ be the coproduct

$$\mathbf{C} = \coprod_{C' \in X^{ss}} C'$$

defined over k. The group $\mathrm{Aut}(\mathbf{C}) = \prod_{C' \in X^{ss}} \mathrm{Aut}(C')$ acts on $\mathbf{C}$ over k. The stack $\mathcal{M}^{ss} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ gives the supersingular points in the formal neighborhood $\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$. Then we have

$$\mathcal{M}^{ss} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p = \coprod_{C' \in X^{ss}} \mathrm{spec}(\overline{\mathbb{F}}_p) /\!/ \mathrm{Aut}(C')$$

$$= \mathrm{stack}\left(k, \prod_{C' \in X^{ss}} \mathrm{Map}(\mathrm{Aut}(C'), \overline{\mathbb{F}}_p)\right).$$

(Here $\mathrm{stack}(-, -)$ denotes the stackification of a Hopf algebroid in the étale topology.) The elliptic curve $\mathbf{C}$ is the pullback of the universal elliptic curve to the cover $\mathrm{spec}(k)$ of $\mathcal{M}^{ss}$.

Let $\widehat{\mathbf{C}}$ be the formal completion of $\mathbf{C}$ at the identity. Let $\mathcal{G}$ be the Lubin-Tate universal deformation of the formal group $\widehat{\mathbf{C}}$ over $\mathbb{W}(k)[[u_1]]$. The induced action of the group $\mathrm{Aut}(\mathbf{C})$ on $\widehat{\mathbf{C}}$ extends to an action on $\mathcal{G}$ over $\mathbb{W}(k)$.

Serre-Tate theory [22], [25] implies that the formal completion functor

$$\{\text{deformations of } \mathbf{C} \text{ over } \mathbb{W}(k)[[u_1]]\}$$

$$\downarrow$$

$$\{\text{deformations of } \widehat{\mathbf{C}} \text{ over } \mathbb{W}(k)[[u_1]]\}$$

is an equivalence of categories. Therefore, there exists a deformation $\widetilde{\mathbf{C}}$ of $\mathbf{C}$ whose formal group is the universal deformation $\mathcal{G}$. Lubin-Tate theory [23] implies that there are no non-trivial automorphisms of the deformation $\mathcal{G}$ which restrict to the identity on $\widehat{\mathbf{C}}$. Therefore, the natural map

$$\mathrm{Aut}(\widetilde{\mathbf{C}}) \xrightarrow{\cong} \mathrm{Aut}(\mathbf{C})$$

is an isomorphism.

The map

$$\chi_{\widetilde{\mathbf{C}}} : \mathrm{spf}(\mathbb{W}(k)[[u_1]]) \to \mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$$

which classifies $\widetilde{\mathbf{C}}$ descends to a map

$$\overline{\chi}_{\widetilde{\mathbf{C}}} : \mathrm{spf}(\mathbb{W}(k)[[u_1]]) /\!/ \mathrm{Aut}(\widetilde{\mathbf{C}}) \to \mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$$

which is an isomorphism. The inverse classifies the universal deformation $\mathcal{G}$.

Now that we have a model for the formal stack $\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$ defined over $\mathbb{W}$, we may use Galois descent to recover the formal stack over $\mathbb{Z}_p$. While the groupoid of $\overline{\mathbb{F}}_p$-points of $\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W}$ is given by supersingular elliptic curves over $\overline{\mathbb{F}}_p$ and isomorphisms which cover the identity on $\overline{\mathbb{F}}_p$, the groupoid of $\overline{\mathbb{F}}_p$-points of $\mathcal{M}^{ss}$ consist of supersingular curves over $\overline{\mathbb{F}}_p$ and isomorphisms which are not required to cover the identity on $\overline{\mathbb{F}}_p$.

In the case of the universal supersingular elliptic curve $\mathbf{C}$, the extra automorphisms arising from the Frobenius may be encoded in an action of the Galois group $Gal = Gal(\overline{\mathbb{F}}_p / \mathbb{F}_p)$ on the groupoid $(\mathrm{spec}(k), \mathrm{Aut}(\mathbf{C}))$. Recall from Section 2.7 that for each curve $C' \in X^{ss}$, there is a Frobenius morphism

$$\mathrm{Frob}_p : C' \to \sigma_* C'$$

where the curve $\sigma_* C'$ is a (possibly different) curve in X^{ss}. Thus there is a map

$$\sigma_* : X^{ss} \to X^{ss}.$$

The action of the generator σ on the objects $\mathrm{spec}(k)$ is given by the composite

$$\sigma^* : k = \prod_{C' \in X^{ss}} \overline{\mathbb{F}}_p \xrightarrow{\text{permute}} \prod_{C' \in X^{ss}} \overline{\mathbb{F}}_p \xrightarrow{\prod \sigma} \prod_{C' \in X^{ss}} \overline{\mathbb{F}}_p = k.$$

The induced map

$$\sigma_* : \mathrm{spec}(k) \to \mathrm{spec}(k)$$

induces the action of Gal on the objects of our groupoid.

The morphisms Frob_p assemble to give an automorphism of $\mathbf{C}$ which covers σ_*.

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\mathrm{Frob}_p} & \mathbf{C} \\
\downarrow & & \downarrow \\
\mathrm{spec}(k) & \xrightarrow[\sigma_*]{} & \mathrm{spec}(k)
\end{array}
$$

The action of σ on the group $\mathrm{Aut}(\mathbf{C})$ is given by conjugation by the automorphism Frob_p of $\mathbf{C}$. We have

$$\sigma_* \alpha = \mathrm{Frob}_p\, \alpha\, \mathrm{Frob}_p^{-1}$$

for each $\alpha \in \mathrm{Aut}(\mathbf{C})$.

There is a profinite Galois covering of formal stacks.

$$
\begin{array}{c}
\mathcal{M}^{ss} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{W} \\
\Big\downarrow {\scriptstyle Gal} \\
\mathcal{M}^{ss}
\end{array}
$$

In a manner completely analogous to Section 2.7, the automorphism group $\mathrm{Aut}(\mathbf{C})$ may be enlarged to include the automorphism Frob_p, giving rise to an extension

$$\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}) = \mathrm{Aut}(\mathbf{C}) \rtimes Gal.$$

5.5. Construction of $TMF_{K(2)}$.

As described in Section 5.1, the Goerss-Hopkins-Miller Theorem gives an E_∞-ring spectrum

$$E(k, \widehat{\mathbf{C}}) \cong \prod_{C' \in X^{ss}} E(\overline{\mathbb{F}}_p, C')$$

and an action of the group $\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C})$ on this spectrum by E_∞-ring maps. The coefficient ring of this complex orientable spectrum is given by

$$E(k, \widehat{\mathbf{C}})_* = \mathbb{W}(k)[[u_1]][u^{\pm 1}]$$

where $|u| = -2$.

The spectrum $TMF_{K(2)}$ is defined to be the homotopy fixed point spectrum

$$TMF_{K(2)} = E(k, \widehat{\mathbf{C}})^{h\,\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C})}$$

$$= \left(\prod_{C' \in X^{ss}} E(\overline{\mathbb{F}}_p, \widehat{C'})^{h\,\mathrm{Aut}(C')} \right)^{hGal}.$$

In Section 6.2 we shall find it useful to work with a version of $TMF_{K(2)}$ where we do not take Galois fixed points. We thus make the definition

$$TMF_{K(2),\overline{\mathbb{F}}_p} = E(k, \widehat{\mathbf{C}})^{h\,\mathrm{Aut}(\mathbf{C})}$$
$$= \prod_{C' \in X^{ss}} E(\overline{\mathbb{F}}_p, \widehat{C}')^{h\,\mathrm{Aut}(C')}.$$

5.6. $\Gamma_0(\ell)$-**structures.** The construction of $TMF_0(\ell)$ is completely analogous. One simply replaces everywhere the formal moduli stack $\mathcal{M}^{ss}$ with $\mathcal{M}_0(\ell)^{ss}$. The automorphism groups $\mathrm{Aut}(C')$ are replaced with $\mathrm{Aut}(C', H)$ for $(C', H) \in X_0^{ss}(\ell)$.

Explicitly, let k' be the perfect ring

$$k' = \prod_{\Gamma_0(\ell)(\mathbf{C})} k = \prod_{X_0^{ss}(\ell)} \overline{\mathbb{F}}_p$$

where $\Gamma_0(\ell)(\mathbf{C})$ is the set of $\Gamma_0(\ell)$-structures on $\mathbf{C}$. We define $\mathbf{C}_0(\ell)$ to be the elliptic curve over k' given by

$$\mathbf{C}_0(\ell) = \coprod_{\Gamma_0(\ell)(\mathbf{C})} \mathbf{C}.$$

We give $\mathbf{C}_0(\ell)$ the canonical $\Gamma_0(\ell)$-structure $\mathbf{H}$ which restricts to H on the component corresponding to the element $H \in \Gamma_0(\ell)(\mathbf{C})$.

Since the map

$$\mathbf{C}[\ell] \to \mathrm{spec}(k)$$

is étale, given a $\Gamma_0(\ell)$-structure H on $\mathbf{C}$, there is a unique extension to a $\Gamma_0(\ell)$-structure $\widetilde{H}$ on $\widetilde{\mathbf{C}}$ over $\mathbb{W}(k)[[u_1]]$. The elliptic curve over $\mathbb{W}(k')[[u_1]]$ given by

$$\widetilde{\mathbf{C}}_0(\ell) = \coprod_{\Gamma_0(\ell)(\mathbf{C})} \widetilde{C}$$

is a deformation of $\mathbf{C}_0(\ell)$. The $\Gamma_0(\ell)$-structure $\mathbf{H}$ extends uniquely to a $\Gamma_0(\ell)$-structure $\widetilde{\mathbf{H}}$ on $\widetilde{\mathbf{C}}_0(\ell)$. It restricts to $\widetilde{H}$ on the component corresponding to the element $H \in \Gamma_0(\ell)(\mathbf{C})$.

Define the group $\mathrm{Aut}(\mathbf{C}_0(\ell), \mathbf{H})$ to be the finite group of automorphisms of $\mathbf{C}_0(\ell)$ which preserve the level structure $\mathbf{H}$:

$$\mathrm{Aut}(\mathbf{C}_0(\ell), \mathbf{H}) = \prod_{(C', H) \in X_0^{ss}(\ell)} \mathrm{Aut}(C', H).$$

The automorphism Frob_p on $\mathbf{C}$ of Section 5.4 will permute the $\Gamma_0(\ell)$-structures, inducing an action of Gal on the groupoid $(\mathrm{spec}(k'), \mathrm{Aut}(\mathbf{C}_0(\ell), \mathbf{H}))$. We get an extension of groups

$$\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H}) = \mathrm{Aut}(\mathbf{C}_0(\ell), \mathbf{H}) \rtimes Gal.$$

Just as in the case of $TMF_{K(2)}$, we use the Goerss-Hopkins-Miller theorem to produce a spectrum $E(k', \widehat{\mathbf{C}}_0(\ell))$. The spectrum $TMF_0(\ell)_{K(2)}$ is given as follows:

$$TMF_0(\ell)_{K(2)} = E(k', \widehat{\mathbf{C}}_0(\ell))^{h\,\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})}$$
$$= \left(\prod_{(C', H) \in X_0^{ss}(\ell)} E(\overline{\mathbb{F}}_p, \widehat{C}')^{h\,\mathrm{Aut}(C', H)} \right)^{hGal}.$$

Just as in the case of TMF, we will define $TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p}$ to be the version where we do not take Galois fixed points:

$$TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p} = E(k', \widehat{\mathbf{C}}_0(\ell))^{h\,\mathrm{Aut}(\mathbf{C}_0(\ell),\mathbf{H})}$$

$$= \prod_{(C',H)\in X_0^{ss}(\ell)} E(\overline{\mathbb{F}}_p, \widehat{C}')^{h\,\mathrm{Aut}(C',H)}.$$

6. Relation to the spectrum $Q(\ell)$

6.1. The spectrum $Q(\ell)$. In [2], using the sheaf $\mathcal{O}_{ell}$ of Section 5.3, we introduced a spectrum $Q(\ell)$ built out of TMF and $TMF_0(\ell)$. We give an independent $K(2)$-local construction here. Nevertheless, the reader might find it useful to refer to [2], where the motivation for the construction is given.

The spectrum $Q(\ell)_{K(2)}$ is the totalization of a semi-cosimplicial E_∞-ring spectrum of the following form:

$$
(6.1.1) \qquad TMF_{K(2)} \quad\Rightarrow\quad
\begin{array}{c}
TMF_0(\ell)_{K(2)} \\
\times \\
TMF_{K(2)}
\end{array}
\quad\Rrightarrow\quad TMF_0(\ell)_{K(2)}.
$$

The coface maps are given in terms of certain maps of E_∞-ring spectra:

$$\phi_f^* : TMF_{K(2)} \to TMF_0(\ell)_{K(2)},$$
$$\phi_q^* : TMF_{K(2)} \to TMF_0(\ell)_{K(2)},$$
$$\psi_{[\ell]}^* : TMF_{K(2)} \to TMF_{K(2)},$$
$$\psi_d^* : TMF_0(\ell)_{K(2)} \to TMF_0(\ell)_{K(2)}.$$

The coface maps on 0-cosimplicies

$$d_i : TMF_{K(2)} \to TMF_0(\ell)_{K(2)} \times TMF_{K(2)}$$

are defined by

$$d_0 = \phi_q^* \times \psi_{[\ell]}^*,$$
$$d_1 = \phi_f^* \times Id.$$

The coface maps

$$d_i : TMF_0(\ell)_{K(2)} \times TMF_{K(2)} \to TMF_0(\ell)_{K(2)}$$

are defined by

$$d_0 = \psi_d^* \circ p_1,$$
$$d_1 = \phi_f^* \circ p_2,$$
$$d_2 = p_1$$

where p_1, p_2 are the projections onto the first and second factors of the product $TMF_0(\ell)_{K(2)} \times TMF_{K(2)}$.

We produce the required maps using the Goerss-Hopkins-Miller functor (Section 5.1).

The map $\psi_{[\ell]}^*$: The ℓth power isogeny

$$[\ell] : \mathbf{C} \to \mathbf{C}$$

induces an automorphism

$$\psi_{[\ell]} = (Id, [\ell]) : (k, \widehat{\mathbf{C}}) \to (k, \widehat{\mathbf{C}}).$$

Applying the Goerss-Hopkins-Miller functor, we get a map

$$\psi_{[\ell]}^* : E(k, \widehat{\mathbf{C}}) \to E(k, \widehat{\mathbf{C}}).$$

Because the ℓ-power isogeny commutes with all of the automorphisms of $\mathbf{C}$, this map descends to the homotopy fixed points

$$\begin{aligned}
\psi_{[\ell]}^* : TMF_{K(2)} &= E(k, \widehat{\mathbf{C}})^{h \, \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C})} \\
&\to E(k, \widehat{\mathbf{C}})^{h \, \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C})} \\
&= TMF_{K(2)}.
\end{aligned}$$

We remark that by replacing the pair $(k, \mathbf{C})$ with the pair $(k', \mathbf{C}_0(\ell))$, we get a map

$$\psi_{[\ell]}^* : TMF_0(\ell)_{K(2)} \to TMF_0(\ell)_{K(2)}.$$

The map ψ_d^*: In Section 3.5 we defined, for each pair $(C', H) \in X_0^{ss}(\ell)$, a pair $(C_H, d(H)) \in X_0^{ss}(\ell)$, and a degree ℓ isogeny

$$\phi_H : C' \to C_H.$$

The isogeny ϕ_H has kernel H, and the dual isogeny $\widehat{\phi}_H$ has kernel $d(H)$. Observe that the pair $(C_H, d(H))$ actually determines (C', H): we have

$$C_{d(H)} = C',$$
$$d(d(H)) = H.$$

We may define an involution

$$\overline{\psi}_d : k' = \prod_{(C', H) \in X_0^{ss}(\ell)} \overline{\mathbb{F}}_p \to \prod_{(C', H) \in X_0^{ss}(\ell)} \overline{\mathbb{F}}_p = k'$$

given by permuting the factors: we send the factor corresponding to (C', H) to the factor corresponding to $(C_H, d(H))$. The maps ϕ_H assemble to give a degree ℓ isogeny

$$\widetilde{\psi}_d : \mathbf{C}_0(\ell) \to \mathbf{C}_0(\ell)$$

which covers the map $\overline{\psi}_d$. The kernel of $\widetilde{\psi}_d$ is $\mathbf{H}$.

Since $\phi_{d(H)}$ and the dual isogeny $\widehat{\phi}_H$ have the same kernel $d(H)$, there exists an automorphism γ_H of C' over $\overline{\mathbb{F}}_p$ so that the following diagram commutes.

(6.1.2)

$$\begin{array}{ccc}
C_H & \xrightarrow{\widehat{\phi}_H} & C' \\
& \searrow{\scriptstyle \phi_{d(H)}} & \downarrow{\scriptstyle \gamma_H} \\
& & C'
\end{array}$$

By applying the dual isogeny functor to the above diagram, we see that γ_H preserves H, so γ_H actually lies in the automorphism group $\mathrm{Aut}(C', H)$. Diagram (6.1.2) gives us the relation

(6.1.3) $$\phi_{d(H)} \circ \phi_H = [\ell] \circ \gamma_H.$$

The automorphisms γ_H assemble to give an automorphism

$$\gamma : \mathbf{C}_0(\ell) \to \mathbf{C}_0(\ell)$$

defined over k' which preserves the subgroup $\mathbf{H}$. Equation (6.1.3) gives us the relation

$$(6.1.4) \qquad \widetilde{\psi}_d \circ \widetilde{\psi}_d = [\ell] \circ \gamma.$$

We assemble these automorphisms to get an automorphism of pairs

$$\psi_d = (\overline{\psi}_d, (\widetilde{\psi}_d)_*) : (k', \widehat{\mathbf{C}}_0(\ell)) \to (k', \widehat{\mathbf{C}}_0(\ell))$$

which induces a map

$$\psi_d^* : E(k', \widehat{\mathbf{C}}_0(\ell)) \to E(k', \widehat{\mathbf{C}}_0(\ell)).$$

We claim that ψ_d^* descends to an automorphism of $TMF_0(\ell)_{K(2)}$. Suppose that β is an element of $\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})$. Then we have the following diagram.

$$(6.1.5) \qquad \begin{array}{ccc} \mathbf{C}_0(\ell) & \xrightarrow{\widetilde{\psi}_d} & \mathbf{C}_0(\ell) \\ \beta \downarrow & & \downarrow \overline{\beta} \\ \mathbf{C}_0(\ell) & \xrightarrow[\widetilde{\psi}_d]{} & \mathbf{C}_0(\ell) \end{array}$$

Since β preserves $\mathbf{H} = \ker \widetilde{\psi}_d$, it descends uniquely to give $\overline{\beta}$. Applying the dual isogeny functor to Diagram (6.1.5), we see that the map $\overline{\beta}$ preserves the kernel of the dual isogeny of $\widetilde{\psi}_d$, But we have argued that this kernel is also given by $\mathbf{H}$. Thus $\overline{\beta}$ also lies in $\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})$. We conclude that the map ψ_d^* descends to the $\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})$-fixed points $TMF_0(\ell)_{K(2)} = E(k', \widehat{\mathbf{C}}_0(\ell))^{h \, \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})}$ to give a map

$$\psi_d^* : TMF_0(\ell)_{K(2)} \to TMF_0(\ell)_{K(2)}.$$

Because γ is contained in $\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})$, it acts trivially on $TMF_0(\ell)_{K(2)}$, and we have the following relation on $TMF_0(\ell)_{K(2)}$.

$$(6.1.6) \qquad \psi_d^* \circ \psi_d^* = \psi_{[\ell]}^*.$$

Remark 6.1.7. The equality in Equation (6.1.6) is a strict equality that occurs on the point-set level. This is because the homotopy fixed point spectrum is the *actual* fixed points of an appropriate fibrant replacement.

The map ϕ_f^*: Let $\overline{\chi}$ denote the diagonal map

$$\overline{\chi} : k \to \prod_{\Gamma_0(\ell)(\mathbf{C})} k = k'.$$

Over this map we have $\mathbf{C}_0(\ell) = \mathbf{C} \otimes_k k'$. We therefore get a map of pairs

$$\chi : (k', \widehat{\mathbf{C}}_0(\ell)) \to (k, \widehat{\mathbf{C}}).$$

The diagonal embedding

$$\mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}) \hookrightarrow \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell))$$

is compatible with the map χ. The natural inclusion

$$\iota : \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H}) \hookrightarrow \mathrm{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell))$$

gives ϕ_f^* as the composite

$$\phi_f^* : TMF_{K(2)} = E(k, \widehat{\mathbf{C}})^{h \operatorname{Aut}_{/\mathbb{F}_p}(\mathbf{C})}$$

$$\xrightarrow{\chi^*} E(k', \widehat{\mathbf{C}}_0(\ell))^{h \operatorname{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell))}$$

$$\xrightarrow{\iota^*} E(k', \widehat{\mathbf{C}}_0(\ell))^{h \operatorname{Aut}_{/\mathbb{F}_p}(\mathbf{C}_0(\ell), \mathbf{H})}$$

$$= TMF_0(\ell)_{K(2)}.$$

The commutativity of the diagram

$$
\begin{array}{ccc}
(k', \widehat{\mathbf{C}}_0(\ell)) & \xrightarrow{\chi} & (k, \widehat{\mathbf{C}}) \\
\psi_{[\ell]} \downarrow & & \downarrow \psi_{[\ell]} \\
(k', \widehat{\mathbf{C}}_0(\ell)) & \xrightarrow{\chi} & (k, \widehat{\mathbf{C}})
\end{array}
$$

implies the relation

(6.1.8) $$\phi_f^* \psi_{[\ell]}^* = \psi_{[\ell]}^* \phi_f^*.$$

The map ϕ_q^*: The map ϕ_q^* is defined to be the composite

$$\phi_q^* : TMF_{K(2)} \xrightarrow{\phi_f^*} TMF_0(\ell)_{K(2)} \xrightarrow{\psi_d^*} TMF_0(\ell)_{K(2)}.$$

The construction of the spectrum $Q(\ell)_{K(2)}$ is completed by the following lemma.

Lemma 6.1.9. The coface maps in (6.1.1) satisfy the cosimplicial identities.

Proof. We translate the cosimplicial identities into the maps that define the d_i's.

(6.1.10) $$d_0 d_0 = d_1 d_0 \qquad \psi_d^* \phi_q^* = \phi_f^* \psi_{[\ell]}^*,$$

(6.1.11) $$d_2 d_0 = d_0 d_1 \qquad \phi_q^* = \psi_d^* \phi_f^*,$$

(6.1.12) $$d_1 d_1 = d_2 d_1 \qquad \phi_f^* = \phi_f^*.$$

Relation (6.1.12) is tautologous, and Relation (6.1.11) is immediate from our definition of ϕ_q^*. Relation (6.1.10) then follows from Relation (6.1.11), Equation (6.1.6), and Equation (6.1.8). □

6.2. $Q(\ell)_{K(2)}$ as the homotopy fixed point spectrum $E(\Gamma)$. In this section we shall prove the following theorem.

Theorem 6.2.1. There is an equivalence $Q(\ell) \simeq E(\Gamma) = (E_2^{h\Gamma_{Gal}})$.

Before we prove these theorems we address some finer points concerning our use of Morava E-theory. We have fixed a supersingular elliptic curve C defined over $\overline{\mathbb{F}}_p$, and have fixed an isomorphism between it and the Honda height 2 formal group H_2 over $\overline{\mathbb{F}}_p$. This gives rise, by the Goerss-Hopkins-Miller theorem, to a fixed isomorphism

$$E(\overline{\mathbb{F}}_p, \widehat{C}) \cong E(\overline{\mathbb{F}}_p, H_2) = E_2.$$

Our fixed isomorphism $\widehat{C} \cong H_2$ also gives an isomorphism

$$\operatorname{Aut}(\overline{\mathbb{F}}_p, \widehat{C}) \cong \operatorname{Aut}(\overline{\mathbb{F}}_p, H_2) = \mathbb{G}_2.$$

In what follows, when we refer to E_2 and $\mathbb{G}_2$, we shall actually be implicitly *identifying* these with $E(\overline{\mathbb{F}}_p, \widehat{C})$ and $\operatorname{Aut}(\overline{\mathbb{F}}_p, \widehat{C})$ using our fixed isomorphisms.

We recall how the Goerss-Hopkins-Miller theorem gives rise to an action of $\mathbb{G}_2$ on E_2 by E_∞-ring maps. Let g be an element of $\mathbb{G}_2$. It is an automorphism

$$g = (g_0, g_1) : (\overline{\mathbb{F}}_p, \widehat{C}) \to (\overline{\mathbb{F}}_p, \widehat{C}).$$

Because the Goerss-Hopkins-Miller theorem gives a *contravariant* functor $E(-,-)$, the left action L_g of g on E_2 is given by the image of g^{-1} under the functor $E(-,-)$

$$(6.2.2) \qquad\qquad L_g = (g^{-1})^* : E_2 \to E_2.$$

For $C' \in X^{ss}$, let $E_{C'}$ denote the spectrum $E(\overline{\mathbb{F}}_p, \widehat{C}')$. We defined $TMF_{K(2)}$ and $TMF_0(\ell)_{K(2)}$ as homotopy fixed points of the spectra

$$E(k, \widehat{\mathbf{C}}) \cong \prod_{C' \in X^{ss}} E_{C'},$$

$$E(k', \widehat{\mathbf{C}}_0(\ell)) \cong \prod_{(C', H) \in X_0^{ss}(\ell)} E_{C'}.$$

The spectra $E_{C'}$ are isomorphic to $E_2 = E_C$ using the fixed isomorphisms (over $\overline{\mathbb{F}}_p$) of formal groups

$$(\phi_{C'})_* : \widehat{C} \to \widehat{C}'$$

induced by the isogenies $\phi_{C'}$ of Section 3.4. We get an induced isomorphism

$$\phi_{C'}^* : E_{C'} \xrightarrow{\cong} E_2.$$

Under this isomorphism, the induced action of the group $\mathrm{Aut}(C')$ on E_2 corresponds to the action given by the embedding $\iota_{C'}$ of $\mathrm{Aut}(C')$ in Γ defined in Section 3.4.

Let $Q(\ell)_{K(2),\overline{\mathbb{F}}_p}$ be the spectrum obtained by the totalization of the Galois equivariant semi-cosimplicial spectrum

$$TMF_{K(2),\overline{\mathbb{F}}_p} \Rightarrow TMF_{K(2),\overline{\mathbb{F}}_p} \times TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p} \Rrightarrow TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p}$$

where we have *not* taken Galois fixed points.

Let $\sigma^{\mathbb{Z}} \subset Gal$ be the discrete group given by powers of the Frobenius σ. The following lemma is a consequence of Theorem 5.2.5.

Lemma 6.2.3. The natural maps

$$TMF_{K(2)} = (TMF_{K(2),\overline{\mathbb{F}}_p})^{hGal} \to (TMF_{K(2),\overline{\mathbb{F}}_p})^{h\sigma^{\mathbb{Z}}},$$

$$TMF_0(\ell)_{K(2)} = (TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p})^{hGal} \to (TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p})^{h\sigma^{\mathbb{Z}}}$$

are equivalences.

Corollary 6.2.4. There is an equivalence

$$Q(\ell)_{K(2)} \to (Q(\ell)_{K(2),\overline{\mathbb{F}}_p})^{h\sigma^{\mathbb{Z}}}.$$

The remainder of this section is devoted to proving Theorem 6.2.1. We shall first prove that there is an equivalence

$$(6.2.5) \qquad\qquad Q(\ell)_{K(2),\overline{\mathbb{F}}_p} \xrightarrow{\simeq} E_2^{h\Gamma}.$$

We will then prove that this equivalence commutes with the action of the Frobenius σ. Theorem 6.2.1 is then recovered by applying the functor $(-)^{h\sigma^{\mathbb{Z}}}$ to (6.2.5).

Proposition 6.2.6. The homotopy fixed point spectrum $E_2^{h\Gamma}$ is the totalization of a semi-cosimplicial spectrum of the form
(6.2.7)

$$\prod_{C'\in X^{ss}} E_2^{h\,\mathrm{Aut}(C')} \Rightarrow \begin{array}{c} \prod_{(C',H)\in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)} \\ \times \\ \prod_{C'\in X^{ss}} E_2^{h\,\mathrm{Aut}(C')} \end{array} \Rrightarrow \prod_{(C',H)\in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)}.$$

The coface maps

$$d_i : \prod_{C'\in X^{ss}} E_2^{h\,\mathrm{Aut}(C')} \to \prod_{(C',H)\in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)} \quad \times \quad \prod_{C'\in X^{ss}} E_2^{h\,\mathrm{Aut}(C')}$$

are defined on components by

$$\begin{aligned}
(d_0)_{(C',H)} &= L_{g_{(C',H)}} \circ \mathrm{Res}_{\mathrm{Aut}(C_H,d(H))}^{\mathrm{Aut}(C_H)}: & E_2^{h\,\mathrm{Aut}(C_H)} &\to E_2^{h\,\mathrm{Aut}(C',H)}, \\
(d_1)_{(C',H)} &= \mathrm{Res}_{\mathrm{Aut}(C',H)}^{\mathrm{Aut}(C')} & : & E_2^{h\,\mathrm{Aut}(C')} \to E_2^{h\,\mathrm{Aut}(C',H)}, \\
(d_0)_{C'} &= L_{\ell^{-1}} & : & E_2^{h\,\mathrm{Aut}(C')} \to E_2^{h\,\mathrm{Aut}(C')}, \\
(d_1)_{C'} &= Id & : & E_2^{h\,\mathrm{Aut}(C')} \to E_2^{h\,\mathrm{Aut}(C')}.
\end{aligned}$$

The coface maps

$$d_i : \prod_{(C',H)\in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)} \quad \times \quad \prod_{C'\in X^{ss}} E_2^{h\,\mathrm{Aut}(C')} \to \prod_{(C',H)\in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)}$$

are defined on components by

$$\begin{aligned}
(d_0)_{(C',H)} &= L_{g_{(C',H)}} & : & E_2^{h\,\mathrm{Aut}(C_H,d(H))} \to E_2^{h\,\mathrm{Aut}(C',H)}, \\
(d_1)_{(C',H)} &= \mathrm{Res}_{\mathrm{Aut}(C',H)}^{\mathrm{Aut}(C')}: & & E_2^{h\,\mathrm{Aut}(C')} \to E_2^{h\,\mathrm{Aut}(C',H)}, \\
(d_2)_{(C',H)} &= Id & : & E_2^{h\,\mathrm{Aut}(C',H)} \to E_2^{h\,\mathrm{Aut}(C',H)}.
\end{aligned}$$

Here the element $g_{(C',H)}$ is the element of Γ defined by Equation (3.5.1).

Proof. Since the complex $\mathcal{J}'$ is non-equivariantly contractible, the natural map

$$E_2^{h\Gamma} \to \mathrm{Map}(\mathcal{J}', E_2)^{h\Gamma}$$

is an equivalence. Using the semi-simplicial structure of $\mathcal{J}'$, we see that $E^{h\Gamma}$ is equivalent to the totalization of the semi-cosimplicial spectrum

(6.2.8) $\mathrm{Map}(\mathcal{J}_0', E_2)^{h\Gamma} \Rightarrow \mathrm{Map}(\mathcal{J}_1', E_2)^{h\Gamma} \Rrightarrow \mathrm{Map}(\mathcal{J}_2', E_2)^{h\Gamma}.$

By "Shapiro's lemma", for a subgroup F of Γ, there is an equivalence

(6.2.9) $E_2^{hF} \simeq \mathrm{Map}(\Gamma/F, E_2)^{h\Gamma}.$

The semi-cosimplicial spectrum given in the proposition is obtained by substituting the descriptions of $\mathcal{J}_i'$ given in Equation (3.4.5) into Equation (6.2.8), and then applying Equation (6.2.9). The descriptions of the coface maps given in the proposition follow immediately from the descriptions of the face maps of $\mathcal{J}_\bullet'$ given by Proposition 3.5.2. $\square$

Construction of the equivalence (6.2.5). We first describe the maps $\psi_{[\ell]}^*$, ψ_d^*, ϕ_f^*, and ϕ_q^* of Section 6.1 under the isomorphisms

$$TMF_{K(2),\overline{\mathbb{F}}_p} \cong \prod_{C' \in X^{ss}} E_{C'}^{h\,\mathrm{Aut}(C')},$$

$$TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p} \cong \prod_{(C',H) \in X_0^{ss}(\ell)} E_{C'}^{h\,\mathrm{Aut}(C',H)}.$$

We describe the components of our maps below, which are read off from their definitions in Section 6.1:

$$\begin{aligned}
(\psi_{[\ell]}^*)_{C'} = [\ell]^* & \quad : \quad & E_{C'}^{h\,\mathrm{Aut}(C')} \to E_{C'}^{h\,\mathrm{Aut}(C')}, \\
(\psi_d^*)_{(C',H)} = \phi_H^* & \quad : \quad & E_{C_H}^{h\,\mathrm{Aut}(C_H,d(H))} \to E_{C'}^{h\,\mathrm{Aut}(C',H)}, \\
(\phi_f^*)_{(C',H)} = \mathrm{Res}_{\mathrm{Aut}(C',H)}^{\mathrm{Aut}(C')} & \quad : \quad & E_{C'}^{h\,\mathrm{Aut}(C')} \to E_{C'}^{h\,\mathrm{Aut}(C',H)}, \\
(\phi_q^*)_{(C',H)} = \phi_H^* \circ \mathrm{Res}_{\mathrm{Aut}(C_H,d(H))}^{\mathrm{Aut}(C_H)} & \quad : \quad & E_{C_H}^{h\,\mathrm{Aut}(C_H)} \to E_{C'}^{h\,\mathrm{Aut}(C',H)}.
\end{aligned}$$

The left $\mathbb{S}_2$ action on E_2 given by Equation (6.2.2) gives the following commutative diagrams.

(6.2.10)

$$\begin{array}{ccc}
E_{C'} \xrightarrow{[\ell]^*} E_{C'} & \qquad & E_{C_H} \xrightarrow{\phi_H^*} E_{C'} \\
\phi_{C'}^* \downarrow \qquad \downarrow \phi_{C'}^* & \qquad & \phi_{C_H}^* \downarrow \qquad \downarrow \phi_{C'}^* \\
E_2 \xrightarrow{L_{\ell-1}} E_2 & \qquad & E_2 \xrightarrow{L_{g_{(C',H)}}} E_2
\end{array}$$

These diagrams, using Proposition 6.2.6, give rise the following equivalence of semi-cosimplicial spectra.
(6.2.11)

The totalization of the top row gives $Q(\ell)_{K(2),\overline{\mathbb{F}}_p}$, while Proposition 6.2.6 implies that the totalization of the bottom row gives $E_2^{h\Gamma}$.

We will finish this section by proving the following lemma.

Lemma 6.2.12. The maps $(\phi_{C'}^*)$ of Diagram 6.2.11 are Galois equivariant.

We pause to explain how Lemma 6.2.12 completes the proof of Theorem 6.2.1. The coface maps of the top row of Diagram 6.2.11 are Galois equivariant by construction. Up to this point, we have not addressed the Galois equivariance of the coface maps of the bottom row of Diagram 6.2.11. By functoriality, the vertical maps $(\phi_{C'}^*)$ are *isomorphisms* of spectra, so the Galois equivariance of the coface

maps of the bottom row will follow from Lemma 6.2.12. The proof of Theorem 6.2.1 is then completed by applying Galois fixed points to the equivalence (6.2.5).

Remark 6.2.13. One could have also deduced the Galois equivariance of the coface maps of the bottom row from the fact that the Galois action on the Tate module $V_\ell(C)$ induces a Galois action on the building $\mathcal{J}'$.

Proof of Lemma 6.2.12. We will only prove that the map

$$(6.2.14) \qquad (\phi_{C'}^*) : \prod_{C' \in X^{ss}} E_{C'}^{h\,\mathrm{Aut}(C')} \to \prod_{C' \in X^{ss}} E_2^{h\,\mathrm{Aut}(C')}$$

of the first column is Galois equivariant. The other case, with level structure, proceeds in the same manner.

We first recall the Galois action on the source and target in Equation (6.2.14). By Equation (6.2.2), the Frobenius σ acts on the source by the map induced by $(\mathrm{Frob}_p^{-1})^*$ (Section 2.7).

$$\sigma : E_{C'}^{h\,\mathrm{Aut}(C')} \xrightarrow{(\mathrm{Frob}_p^{-1})^*} E_{\sigma_*C'}^{h\,\mathrm{Aut}(\sigma_*C')}.$$

The Frobenius action on the target in Equation (6.2.14) is induced from the Frobenius action on the Γ-set $\coprod_{X^{ss}} \Gamma/\phi_{C'}^{-1}\mathrm{Aut}(C')\phi_{C'}$ (as described by Equation (3.4.6)) through our application of Shapiro's lemma (Equation (6.2.9)). The resulting Frobenius action is given by

$$\sigma : E_2^{h\,\mathrm{Aut}(C')} \xrightarrow{y_{C'}^* \circ (\mathrm{Frob}_p^{-1})^*} E_2^{h\,\mathrm{Aut}(\sigma_*C')}$$

where the quasi-isogeny $y_{C'}$ is defined in the proof of Theorem 3.4.4. The lemma now follows from the commutativity of the following diagram, which is immediate given the definition of $y_{C'}$.

$$\begin{CD}
E_{C'}^{h\,\mathrm{Aut}(C')} @>{(\mathrm{Frob}_p^{-1})^*}>> E_{\sigma_*C'}^{h\,\mathrm{Aut}(\sigma_*C')} \\
@V{\phi_{C'}^*}VV @VV{\phi_{\sigma_*C'}^*}V \\
E_2^{h\,\mathrm{Aut}(C')} @>>{y_{C'}^* \circ (\mathrm{Frob}_p^{-1})^*}> E_2^{h\,\mathrm{Aut}(\sigma_*C')}
\end{CD}$$

$\square$

6.3. A resolution of $E_2^{h\Gamma_{Gal}^1}$.

In this section we explain how the statement of Theorem 6.2.1 changes when we replace the spectrum $E_2^{h\Gamma_{Gal}}$ with the spectrum $E_2^{h\Gamma_{Gal}^1}$, where Γ_{Gal}^1 is the norm 1 subgroup defined in Section 2.7.

Theorem 6.3.1. The spectrum $E_2^{h\Gamma_{Gal}^1}$ is equivalent to the homotopy fiber of the map

$$TMF_{K(2)} \times TMF_{K(2)} \xrightarrow{p_2 \circ \phi_q^* - p_1 \circ \phi_f^*} TMF_0(\ell)_{K(2)}$$

where the maps p_i are projections and the maps ϕ_q^* and ϕ_f^* are the maps defined in Section 6.1.

The proof of Theorem 6.3.1 follows the same lines as the proof of Theorem 6.2.1. Namely, one uses Equation 4.4.1 to deduce the analog of Proposition 6.2.6: the

spectrum $E_2^{h\Gamma^1}$ is equivalent to the totalization of a semi-cosimplicial spectrum of the form

$$\prod_{C' \in X^{ss}} E_2^{h\,\mathrm{Aut}(C')} \times E_2^{h\overline{\mathrm{Aut}(C')}} \rightrightarrows \prod_{(C',H) \in X_0^{ss}(\ell)} E_2^{h\,\mathrm{Aut}(C',H)}.$$

One then forms the analog of Diagram 6.2.11

(6.3.2)
$$
\begin{array}{ccc}
\prod_{C'} E_{C'}^{h\,\mathrm{Aut}(C')} \times E_{C'}^{h\,\mathrm{Aut}(C')} & \Longrightarrow & \prod_{(C',H)} E_{C'}^{h\,\mathrm{Aut}(C',H)} \\
\downarrow {\scriptstyle (\phi_{C'}^*) \times ((\ell^{-r}\phi_{C'}\phi)^*)} & & \downarrow {\scriptstyle (\phi_{C'}^*)} \\
\prod_{C'} E_2^{h\,\mathrm{Aut}(C')} \times E_2^{h\overline{\mathrm{Aut}(C')}} & \Longrightarrow & \prod_{(C',H)} E_2^{h\,\mathrm{Aut}(C',H)}
\end{array}
$$

where the coface maps of the top row correspond to $p_2 \circ \phi_q^*$ and $p_1 \circ \phi_f^*$, and the coface maps of the bottom row are determined by Proposition 4.4.2. Here ϕ is the endomorphism of C of degree ℓ^{2r+1} that we chose in Section 4.3. The essential point to the commutativity of Diagram 6.3.2 is the analog of Diagram 6.2.10: for each (C',H), the following diagram commutes.

$$
\begin{array}{ccc}
E_{C_H} & \xrightarrow{\phi_H^*} & E_{C'} \\
\downarrow {\scriptstyle (\ell^{-r}\phi_{C_H}\phi)^*} & & \downarrow {\scriptstyle \phi_{C'}^*} \\
E_2 & \xrightarrow{L_{g^1_{(C',H)}}} & E_2
\end{array}
$$

We deduce that there is an equivalence between $E_2^{h\Gamma^1}$ and the homotopy fiber of the map

$$TMF_{K(2),\overline{\mathbb{F}}_p} \times TMF_{K(2),\overline{\mathbb{F}}_p} \xrightarrow{p_2 \circ \phi_q^* - p_1 \circ \phi_f^*} TMF_0(\ell)_{K(2),\overline{\mathbb{F}}_p}.$$

The Galois equivariance of this equivalence follows the same line of verification that appears in the proof of Lemma 6.2.12, and thus Theorem 6.3.1 is obtained by taking Galois homotopy fixed points.

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