Orientations and Eisenstein Series

Note Title 9/14/2008

I. Orientations

Notation R = assoc. ring spectrum

$$\begin{tikzcd} GL_1 R \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\ \pi_0 R^\times \arrow[r] & \pi_0 R \end{tikzcd}$$

$GL_1 R$ is a gp-like top'l monoid

R = commutative ring spectrum

$\Rightarrow$ $GL_1 R$ is a gp-like E_∞-space

$\Rightarrow$ $GL_1 R = \Omega^\infty gl_1 R$ $\quad$ $gl_1 R$ = connective spectrum

G = top'l gp

$G \longrightarrow GL_1 S$ map of top'l monoids

$$\begin{tikzcd} G_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & (\Omega^n S^n)^\times \arrow[d] \\ G \arrow[r] & GL_1 S \end{tikzcd}$$

$G_n \circlearrowright S^n$

$G = \lim G_n$

X = space

Given principle G-bundle

$$\begin{tikzcd} E \arrow[r] \arrow[d, "G"] & EG \arrow[d] \\ X \arrow[r, "\xi"'] & BG \end{tikzcd}$$

$X = \operatorname{colim} X_i$

finite CW complexes

$$\begin{tikzcd} & BG_{n_i} \arrow[d] \\ X_i \arrow[r] \arrow[ur, dashed] & BG \end{tikzcd}$$

$\{n_i\}$ increasing

$n_i \xrightarrow[i \to \infty]{} \infty$

Form:

$$X_{i+} \wedge_{G_{n_i}} S^{n_i}$$

$$\Sigma^{n_{i+1} - n_i}\left(X_{i+} \wedge_{G_{n_i}} S^{n_i}\right) \cong X_{i+} \wedge_{G_{n_{i+1}}} S^{n_{i+1}} \longrightarrow (X_{i+1})_+ \wedge_{G_{n_{i+1}}} S^{n_{i+1}}$$

Define: Thom Spectrum

$$X^{\xi} = \varinjlim \Sigma^{-n_i} \Sigma^{\infty} X_{i+} \wedge_{G_{n_i}} S^{n_i}$$

Universal case

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG \arrow[d] \\ BG \arrow[r, equal, "\xi_{univ}"'] & BG \end{tikzcd} \quad \rightsquigarrow \quad MG := (BG)^{\xi_{univ}}$$

Suggestive notation: (from thy of parametrized spectra)

$$X^{\xi} =: E_+ \wedge_G S$$

$$MG =: EG_+ \wedge_G S$$

MG is a ring spectrum.

If $G = \Omega^\infty g$ $\quad g$ = connective spectrum

and $G \longrightarrow GL_1 S$ is Ω^∞-map

(i.e. arises from $g \longrightarrow gl_1 S$)

Then MG = E_∞-ring spectrum.

Multiplicative orientation

Map of Ring spectra

$$MG \xrightarrow[\Phi]{} R$$

E_∞-orientation

Map of E_∞-rings

$$MG \xrightarrow[\Phi]{} R$$

E_∞-orientation theory

Def: The E_∞-orientation obstruction for R is the map

$$ob_R : g \longrightarrow gl_1 S \longrightarrow gl_1 R$$

Thm:

(i) $\left\{ \begin{array}{l} \exists \;\; E_\infty\text{-orientation} \\ \quad MG \longrightarrow R \end{array} \right\} \iff \{ ob_R = 0 \}$

(ii) Let C_g be the cofiber

$$g \longrightarrow gl_1 S \longrightarrow C_g$$

Then there is an equivalence of spaces

$$E_\infty(MG, R) \simeq \left\{ \begin{array}{c} \text{Space of factorizations} \\ \begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_g \arrow[dl, dashed] \\ gl_1 R & \end{tikzcd} \end{array} \right\}$$

Note: this implies that there is a homotopy pullback:

$$\begin{tikzcd} E_\infty(MG, R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Map}}(C_g, gl_1 R) \arrow[d] \\ * \arrow[r, "\text{unit}"'] & \underline{\mathrm{Map}}(gl_1 S, gl_1 R) \end{tikzcd}$$

In particular:

$$\pi_0 E_\infty(MG, R) \twoheadrightarrow \left\{ \begin{array}{c} \text{factorizations in} \\ \text{homotopy cat} \\ \begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_g \arrow[dl, dashed, "\phi"] \\ gl_1 R & \end{tikzcd} \end{array} \right\}$$

(torsor for a quotient of $\pi_1 \underline{\mathrm{Map}}(gl_1 S, gl_1 R)$)

<u>Problem</u>: compute set of factorizations:

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_g \arrow[dl, dashed] \\ gl_1 R & \end{tikzcd}$$

(if its empty, $ob_\Phi \neq 0$)

II.) Homotopy type of $gl_1 R$

$R =$ $E(n)$-local

(1) Understand $(gl_1 R)_{K(i)}$ $0 \leq i \leq n$

using Rezk's Logarithm + 'Hecke operators'

(2) Understand $(gl_1 R)_{E(n)}$

using Chromatic fracture and "Adams operators"

(3) $d_n R \longrightarrow gl_1 R \longrightarrow (gl_1 R)_{E(n)}$

"discrepancy spectrum"

<u>prove</u>: $\pi_{>n+1} d_n R = 0$

<u>Using</u>: $\widetilde{K(n)}_* K(\mathbb{Z}/p^i, q) = 0$, $q > n$

$\widetilde{K(n)}_* K(\mathbb{Z}, q) = 0$, $q > n+1$

II.1 Understanding $(gl_1 R)_{K(i)}$

Bousfield-Kuhn:

$$\begin{tikzcd} Sp \arrow[rr, "(-)_{K(i)}"] \arrow[dr, "\Omega^\infty"'] & & Sp \\ & Top_* \arrow[ur, "\Phi_i"'] & \end{tikzcd}$$

$$\begin{tikzcd} gl_1 R \arrow[r] \arrow[rrr, bend right, "\log_i"'] & (gl_1 R)_{K(i)} \simeq \Phi_i(GL_1(R)) \arrow[r] & \Phi_i(\Omega^\infty R) \simeq & R_{K(i)} \end{tikzcd}$$

Rezk $R = E_\infty$ even periodic orientable + unspecified hypotheses

Note: $\pi_k gl_1 R \cong \begin{cases} \pi_0 R^\times & k = 0 \\ \pi_k R & k > 0 \end{cases}$

Compute: $(\log_i)_*: \pi_k(R) \longrightarrow \pi_k(R)$

$k > 0$

Power operations

$$P_{p^i} : R \longrightarrow R^{B\Sigma_{p^i +}}$$

Strickland

$$\pi_* R^{B\Sigma_{p^i +}} \twoheadrightarrow \mathcal{O}_{Sub_{p^i}(G_R)}$$

Here $Sub_{p^i}(G_R)$ = Scheme of subgps of order p^i in G_R

Now: up to flat extension, if ht $\mathbb{G} = n$

$$\mathrm{Sub}_{p^j}(\mathbb{G}) \cong \left\{ H \le (\mathbb{Z}/p^\infty)^n \;\middle|\; |H| = p^j \right\}$$

$R \longrightarrow S$

Given a subgp $H \le (\mathbb{G}_R)_S$, $|H| = p^j$

power operation:

$$\begin{tikzcd}
\pi_* R \arrow[r, "P_j"'] \arrow[rrr, bend right, "\psi_H"'] & \pi_* R^{B\Sigma_{p^j}} \arrow[r] & \mathcal{O}_{\mathrm{Sub}_{p^j}(\mathbb{G}_R)} \arrow[r] & S
\end{tikzcd}$$

classifies H

"quotient $\mathbb{G} \longrightarrow \mathbb{G}/H$"

Rezk's formula

$$\mathbb{G}_R^{(i)} = \mathbb{G}_{R_{K(i)}} / \mathrm{Spf}(\pi_0 R_{K(i)})$$

formal gp ht i

$$\pi_* R \xrightarrow[(\log_i)_*]{} \pi_* R_{K(i)}$$

$$\log_i(x) = \sum_{j=0}^{i} (-1)^j p^{\frac{j(j-1)}{2}} \overbrace{\frac{1}{p^j} \sum_{\substack{H \le \mathbb{G}_R^{(i)}[p] \\ |H| = p^j}} \psi_H(x)}^{\text{"Hecke operator" } T_{p^j}}$$

By def: $\log_i$ gives an equivalence:

$$gl_1(R)_{K(i)} \xrightarrow[\simeq]{\log_i} R_{K(i)}$$

Case $i=0$

$$x \longmapsto x$$

$$gl_1(R)_{\mathbb{Q}} \longrightarrow R_{\mathbb{Q}}$$

iso on $\pi_{>0}$

II.2 $gl_1(R)_{E(n)}$: chromatic fracture + Atkin operators

X = spectrum

$I = \{n_1 < n_2 < \cdots < n_k\} \subset \underline{n} = \{0, \ldots, n\}$

$$X_{K(I)} := \left(\cdots\left(X_{K(n_k)}\right)_{K(n_{k-1})} \cdots\right)_{K(n_1)}$$

Chromatic Fracture

$$X_{E(n)} \simeq \underset{\phi \neq I \subseteq \underline{n}}{\overleftarrow{\operatorname{holim}}}\, X_{K(I)}$$

$$(gl_1 R)_{E(n)} \simeq \underset{0 \le n_1 < \cdots < n_k \le n}{\overleftarrow{\operatorname{holim}}} (gl_1 R)_{K(n_k, \ldots, n_1)}$$

$$\downarrow \overleftarrow{\operatorname{holim}} (\log_{n_k})_{K(n_{k-1}, \ldots, n_1)}$$

$$\underset{0 \le n_1 < \cdots < n_k \le n}{\overleftarrow{\operatorname{holim}}} R_{K(n_k, \ldots, n_1)}$$

Modulo a π_0 issue coming from $\log_0$

$$(gl_1 R)_{\Sigma(n)} \simeq \underset{0 \le n_1 < \cdots < n_k \le n}{\operatorname{holim}} R_{K(n_k, \ldots, n_1)} \qquad (*)$$

Q: what are the maps in this holim

Atkin operations

$$\mathbb{G}_R = \underset{\overrightarrow{k}}{\operatorname{holim}} \operatorname{Spec}\left(\pi_0 R^{B\mathbb{Z}/p^k_+}\right) \qquad \text{formal gp}$$

$$\pi_0 R_{K(i)} = \text{complete wrt } \text{``}(\underbrace{p, v_1, \ldots, v_i}_{I_i})\text{''}$$

$$\rightsquigarrow \underbrace{\operatorname{Spf}_{I_i}(\pi_0 R_{K(i)})}_{= X^{(i)}} \supset \underbrace{\operatorname{Spec}\left(\pi_0 R_{K(i)}/I_i\right)}_{= X_0^{(i)}}$$

special fiber

$$X = \operatorname{Spec}(\pi_0 R)$$

$$\mathbb{G}_R\Big|_{X_0^{(i)}} \text{ has constant height } i$$

$$\Rightarrow \mathbb{G}_R\Big|_{X^{(i)}} = \mathbb{G}_R^{(i)} \text{ is an infinitesimal } p\text{-divisible gp of ht } i$$

$$\mathbb{G}_R^{(i)} = \underset{\overrightarrow{k}}{\lim} \operatorname{Spf}_{I_i}\left(\pi_0 (R_{K(i)})^{B\mathbb{Z}/p^k_+}\right)$$

$$\tilde{X}^{(i)} = \mathrm{Spec}\left(\pi_0 R_{K(i)}\right)$$

"Algebization" $\Longrightarrow$ $\tilde{\mathbb{G}}^{(i)}_R$ a p-divisible gp of ht i

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$j < i$

$\pi_0 \left(R_{K(i)}\right)_{K(j)}$

complete w.r.t. I_j

$$X^{(i,j)} := \mathrm{Spf}_{I_j}\left(\pi_0 R_{K(i,j)}\right)$$

p-divisible gp

$$\mathbb{G}^{(i,j)}_R := \tilde{\mathbb{G}}^{(i)}\Big|_{X^{(i,j)}} \qquad \text{ht} = i$$

$$\begin{array}{ccccc} \text{ht } j & & \text{ht } i & & \text{ht } i-j \\ \left(\mathbb{G}^{(i,j)}_R\right)_{inf} & \longrightarrow & \mathbb{G}^{(i,j)}_R & \longrightarrow & \left(\mathbb{G}^{(i,j)}_R\right)_{et} \end{array}$$

$$\mathbb{G}^{(i,j)}_R = \varinjlim_k \mathrm{Spf}_{I_j}\left(\left(\left(R_{K(i)}\right)^{B\mathbb{Z}/p^k_+}\right)_{K(j)}\right)$$

$$\left(\mathbb{G}^{(i,j)}_R\right)_{inf} = \varinjlim_k \mathrm{Spf}_{I_j}\left(\left(R_{K(i,j)}\right)^{B\mathbb{Z}/p^k_+}\right)$$

The diagram $(*)$ is "generated" by

maps $j < i$

$$\log_i^j : R_{K(i)} \longrightarrow R_{K(i,j)}$$

In general: the maps "edges of chromatic fracture cube"

are given by:

$$R_{K(n_k, n_{k-1}, \ldots, n_1)} \xrightarrow[\left(\log_{n_k}^{m}\right)_{K(n_{k-1}, \ldots, n_1)}]{} R_{K(m, n_k, n_{k-1}, \ldots, n_1)}$$

or, for $t \leq k$

$$R_{K(n_k, \ldots, n_t, n_{t-1}, \ldots, n_1)} \xrightarrow[\left(\alpha^{m}_{R_{K(n_k, \ldots, n_t)}}\right)_{K(n_{t-1}, \ldots, n_1)}]{} R_{K(n_k, \ldots, n_t, m, n_{t-1}, \ldots, n_1)}$$

(where $\alpha_X^m : X \longrightarrow X_{K(m)}$)

Pre Thm: the map $(\log_i^j)_*\, \pi_* R_{K(i)} \longrightarrow \pi_* R_{K(i,j)}$

is given by:

$$\log_i^j(x) = \sum_{k=0}^{j-i} (-1)^k p^{\frac{k(k-1)}{2}} \frac{1}{p^k} \sum_{\substack{H \leq \mathbb{G}_R^{(i,j)}[p] \\ |H| = p^k \\ H \cap (\mathbb{G}_R^{(i,j)})_{et} = 0}} \psi_H(x)$$

Key idea:

There are factorizations

$$\begin{tikzcd}
\pi_*(gl_1 R) \arrow[rr, "\log_j"] \arrow[dr, "(\log_i)_{K(j)}"'] & & \pi_* R_{K(j)} \arrow[dl, dashed, "\log_i^j"] \\
& \pi_* R_{K(i,j)} &
\end{tikzcd}$$

Example (Ando-Hopkins-Rezk)

$$\begin{tikzcd}
(gl_1 \mathrm{tmf})_{K(1)\vee K(2)} \arrow[r, "\log_2"] \arrow[d, "\log_1"] & \mathrm{tmf}_{K(2)} \arrow[d, "\alpha'_{\mathrm{tmf}_{K(2)}}"] \\
\mathrm{tmf}_{K(1)} \arrow[r, "1-U_p"'] & \mathrm{tmf}_{K(2),K(1)}
\end{tikzcd}$$

In terms of modular forms:

$$\log_2 = 1 - T_p + p^{k-1} \qquad \text{(on weight } k\text{)}$$

$$= 1 - \frac{1}{p}\sum_{\substack{H \le C[p] \\ |H| = p}}' \psi_H + \frac{1}{p}\psi_{C[p^2]} \qquad C = \text{elliptic curve}$$

C = ordinary elliptic curve:

$$\log_1 = 1 - p^{k-1}V_p = 1 - \frac{1}{p}\psi_{C[p]_{inf}}$$

$$1 - U_p = 1 - \frac{1}{p}\sum_{\substack{H \subset C[p] \\ H \cap C[p]_{inf} = 0}}' \psi_H \qquad |H| = p$$

III Rational orientations and Hirzebruch series

We consider the case: $BG_N = BO\langle N \rangle$

so $G_N = O\langle N-1 \rangle$

e.g. $G_4 = \mathrm{Spin}$

$G_8 = \mathrm{String}$

Suppose $R = R_{\mathbb{Q}}$

$$\Rightarrow (gl_1 R)\langle 1 \rangle \xrightarrow[\log_0]{\simeq} R\langle 1 \rangle$$

$$(gl_1 S)_{\mathbb{Q}} \simeq *$$

$$[g_N, gl_1 R] \simeq [g_N, gl_1 R\langle 1 \rangle] \simeq [g_N, R_{\mathbb{Q}}]$$

$$\begin{tikzcd} g_N \arrow[r] & gl_1 S \arrow[r] \arrow[d] & C_{g_N} \arrow[dl, dashed, "f"] \\ & gl_1(R) & \end{tikzcd}$$

$$\Updownarrow$$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & (bg_N)_{\mathbb{Q}} \arrow[dl, dashed, "f"] \\ R_{\mathbb{Q}} & \end{tikzcd}$$

Note: $bg_N \cong bo\langle N \rangle$

$$\pi_*(bo\langle N\rangle)_{\mathbb{Q}} \hookrightarrow \pi_* KU_{\mathbb{Q}}$$

$$\mathbb{Q}\{u^{2t}, u^{2t+2}, u^{2t+4}, \dots\} \subset \mathbb{Q}[u, u^{-1}] \qquad |u| = 2$$

$$t := \lceil N/4 \rceil$$

---

f is determined by: $\qquad f(u^{2k}) = b_{2k} \in \pi_{4k} R_{\mathbb{Q}}$

Set $b_{2k+1} := 0$

Thm: (Ando–Hopkins–Rezk)

$$K_\Phi(x) = \exp\Big(\sum_{k \geq 2t} t_k \frac{x^k}{k!}\Big)$$

$$t_k = 2b_k$$

$$\Phi(M) = \langle K_\Phi(TM), [M] \rangle$$

---

e.g. $\quad \Phi = \hat{A} \qquad t_k = \frac{-B_k}{k} = \zeta(1-k) \qquad k \geq$

$N = 4$

$\Phi = $ Witten $\qquad t_k = 2G_k \qquad q = e^{2\pi i \tau}$

$N = 8$

$$2G_k(\tau) = \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \cancel{SL_2(\mathbb{Z})\cap P} \\ SL_2(\mathbb{Z})}} \frac{1}{(c\tau + d)^k} = \frac{-B_k}{k} + 2\sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

IV – E(1)-local orientations + p-adic L-functions

Assume: $N \geq 4$, $R \simeq R_{E(1)}$, p

$$\Rightarrow \quad \pi_{\leq 2}\, g_N = 0$$

Since $\pi_{>2}\, d_1 R = 0$

$$[g_N, d_1 R] \longrightarrow [g_N, gl_1 R] \longrightarrow [g_N, (gl_1 R)_{E(1)}]$$

$\cong$

$*$

So $\quad ob_R = 0$

$\iff (ob_R)_{E(1)} = 0$

Want to understand

$$\begin{tikzcd}
C g_N \arrow[dr, dashed, "\delta"] \arrow[drr, bend left, "\delta_1"] \arrow[ddr, bend right, "\delta_0"'] & & \\
& gl_1(R)_{E(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & gl_1(R)_{K(1)} \arrow[d] \\
& gl_1(R)_{\mathbb{Q}} \arrow[r] & gl_1(R)_{K(1),\mathbb{Q}}
\end{tikzcd}$$

Identify!

$$\boxed{[C_{g_N},\ gl_1(R)_{K(1)}]}$$

$$\langle \ell \rangle = \mathbb{Z}_p^\times/(\pm 1)$$

Thm! There is a map of fiber sequences

$$\begin{tikzcd}
(g_N)_{K(1)} \arrow[r] & (gl_1 S)_{K(1)} \arrow[r] & (C_{g_N})_{K(1)} \arrow[r] & (bo\langle N\rangle)_{K(1)} \\
\Sigma^{-1} KO_p \arrow[r] \arrow[u, "\simeq"] & S_{K(1)} \arrow[r] \arrow[u, "\simeq"] & KO_p \arrow[r, "\psi^\ell - 1", "\simeq"'] \arrow[u, dashed, "\alpha", "\simeq"'] & KO_p \arrow[u, "\simeq"]
\end{tikzcd}$$

(pf) The unstable Adams conjecture implies that the composite:

$$\begin{tikzcd}
BG_N \arrow[r, "\psi^\ell - 1"'] \arrow[rr, bend right, "*"'] & BG_N \arrow[r] & BGL_1 S
\end{tikzcd}$$

Applying Φ_1 gives

$$\begin{tikzcd}
KO_p \arrow[r, "\psi^\ell - 1"'] \arrow[rr, bend right, "*"'] & KO_p \arrow[r] & (bgl_1 S)_{K(1)}
\end{tikzcd}$$

Thus we have:

$$\begin{tikzcd}
KO_p \arrow[r, "\psi^\ell - 1"] \arrow[d, dashed, "\alpha"'] & KO_p \arrow[r] \arrow[d, "\simeq"] & \Sigma S_{K(1)} \arrow[d, dashed, "\simeq"] \\
C_{g_N} \arrow[r] & (bg_N)_{K(1)} \arrow[r] & (bgl_1 S)_{K(1)}
\end{tikzcd}$$

Thus

$$
\begin{tikzcd}
{[C_{gN},\ (gl_1 R)_{K(1)}]} \arrow[r, "\cong", "\alpha^*"'] & {[KO_p,\ (gl_1 R)_{K(1)}]} \arrow[d, "(\log_1)_*"] \\
 & {[KO_p,\ R_{K(1)}]}
\end{tikzcd}
$$

use fiber sequence:

$$R_{K(1)} \longrightarrow (KO \wedge R)_{K(1)} \longrightarrow (KO \wedge R)_{K(1)}$$

and

$$[KO_p,\ (KO\wedge R)_{K(1)}] \simeq [(KO \wedge KO)_{K(1)},\ (KO\wedge R)_{K(1)}]_{KO_p}$$

assume this is torsion free

Let $\mathbb{V}_R := \pi_0 (KO \wedge R)_{K(1)}$ "big p-adic space"

Assume $\pi_* KO \wedge R \simeq \mathbb{V}_R \otimes KO_*$

$\mathbb{V}_R$ carries action by $\mathbb{Z}_p^\times/\{\pm 1\}$

"Adams-Diamond operators" ψ^λ

Assume $H_c^*(\mathbb{Z}_p^\times/\{\pm1\}, \mathbb{V}_R) = 0 \quad * > 0$

Then $\pi_{4k} R_{K(1)} \cong \mathbb{V}_R\langle 2k\rangle := \{v \in \mathbb{V}_R \mid \psi^\lambda v = \lambda^{2k}\cdot v\}$

$$(KO \wedge KO)_{K(1)} \simeq \mathrm{Map}^c\left(\mathbb{Z}_p^\times/\{\pm 1\}, KO_*\right)$$

$\Longrightarrow$ $$\left[(KO_p \wedge KO_p)_{K(1)}, (KO_p \wedge R)_{K(1)}\right]_{KO}$$

universal coefficient SS

$$\cong$$

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Map}^c(\mathbb{Z}_p^\times/(\pm 1), \mathbb{Z}_p), \mathbb{V}_R\right)$$

$$\cong$$

$\mathbb{V}_R$ - valued measures on $\mathbb{Z}_p^\times/(\pm 1)$

$\mathrm{Meas}(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p)$

---

given $\mu \in \mathrm{Meas}(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p)$

$f \in \mathrm{Map}^c(\mathbb{Z}_p^\times/(\pm 1), \mathbb{Z}_p)$

<u>get</u> $$\int_{\mathbb{Z}_p^\times/\pm 1} f \, d\mu \in \mathbb{V}_R$$

determined by moments!

$$\int_{\mathbb{Z}_p^\times/\pm 1} x^{2k} \, d\mu \in \mathbb{V}_R$$

Our assumption on $H_c^*(\,;\, \mathbb{V}_R)$

<u>implies</u>:

Lem with our many assumptions:

$$[KO_p, R_{K(1)}] \hookrightarrow \mathrm{Meas}\left(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{V}_R\right)$$

$$\shortparallel$$

$$\left\{ \mu \;\middle|\; \int_{\mathbb{Z}_p^\times/\pm 1} x^{2k}\, d\mu \in \mathbb{V}_R\langle 2k\rangle \right\}$$

Thm:

$$[C_{g_N}, (gl_1 R)_{E(1)}] \cong \left\{ \left(\{b_{2k}\}_{2k \geq t}, \mu \right) \right\}$$

$$b_{2k} \in \pi_{4k} R_{\mathbb{Q}} \longrightarrow \pi_{4k}(R_{K(1)})_{\mathbb{Q}} \cong \mathbb{V}_R\langle 2k\rangle \otimes \mathbb{Q}$$

$$\mu \in \mathrm{Meas}\left(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{V}_R\right)$$

s.t. $$\int_{\mathbb{Z}_p^\times/\pm 1} x^{2k}\, d\mu = (\ell^{2k} - 1)\, b_{2k}^*$$

$$\left(\text{in } \mathbb{V}_R \otimes \mathbb{Q} \right)$$

where $b_{2k}^* := \log_1(b_{2k})$

(pf)

$$
\begin{tikzcd}
C_{g_N} \arrow[dr, dashed] \arrow[drr, bend left, "\mu"] \arrow[ddr, bend right, "\{b_{2k}\}"'] & & & \\
& (gl_1 R)_{E(1)} \arrow[r] \arrow[d] & (gl_1 R)_{K(1)} \arrow[r, "\log_1"] \arrow[d] & R_{K(1)} \arrow[dd] \\
& (gl_1 R)_{\mathbb{Q}} \arrow[r] \arrow[d, "\log_0"] & (gl_1 R)_{K(1),\mathbb{Q}} \arrow[dr, "(\log_1)_{\mathbb{Q}}"] & \\
& R_{\mathbb{Q}} \arrow[rr, "(\log_1)_{\mathbb{Q}}"'] & & R_{K(1),\mathbb{Q}}
\end{tikzcd}
$$

<u>Using</u>:

$$
(gl_1 S)_{\mathbb{Q}} \longrightarrow (C_{g_N})_{\mathbb{Q}} \xrightarrow[\beta]{\simeq} (bo\langle N\rangle)_{\mathbb{Q}}
$$

<u>And</u>:

$$
\begin{tikzcd}
(gl_1 S)_{K(1)} \arrow[r] & (C_{g_N})_{K(1)} \arrow[r, "\beta"] & (bo\langle N\rangle)_{K(1)} \\
S_{K(1)} \arrow[r] \arrow[u, "\simeq"'] & KO_p \arrow[r, "\psi^{\ell}-1"'] \arrow[u, dashed, "\alpha"'] & KO_p \arrow[u, "\simeq"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \{b_{2k}\} \\
{[C_{g_N}, (gl_1 R)_{\mathbb{Q}}]} \arrow[r, "\cong", "(\log_0)_*"'] \arrow[d] & {[C_{g_N}, R_{\mathbb{Q}}]} \arrow[d, "(\log_1)_*"] & {[bo\langle N\rangle_{\mathbb{Q}}, R_{\mathbb{Q}}]} \arrow[l, "\cong"', "\beta^*"] \arrow[d, "(*)"] \arrow[u] \\
{[C_{g_N}, (gl_1 R)_{K(1),\mathbb{Q}}]} \arrow[r, "(\log_1)_*"'] & {[C_{g_N}, R_{K(1),\mathbb{Q}}]} \arrow[r, "\cong", "\alpha^*"'] & {[KO_p, R_{K(1),\mathbb{Q}}]} \\
{[C_{g_N}, (gl_1 R)_{K(1)}]} \arrow[r] \arrow[u] & {[C_{g_N}, R_{K(1)}]} \arrow[r] \arrow[u] & {[KO_p, R_{K(1)}]} \arrow[u]
\end{tikzcd}
$$

$\ni$

μ

$$\text{“}(\log_1)_* \alpha^* \beta^* \text{”}$$

$$\|$$

We see that under $(*)$, $\{b_{2k}\}$ maps to a measure w/ moments

$$\int x^{2k} = (\ell^{2k}-1)\, b_{2k}$$

//

We have identified $[C_{g_N}, (gl_1 R)_{E(1)}]$

Now just need to pick out those for which

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_{g_N} \arrow[dl, dashed] \\ (gl_1 R)_{E(1)} & \end{tikzcd}$$

commutes.

Miller Invariants

Define $R_{\mathbb{Q}/S}$ to be the cofiber

$$R \longrightarrow R_{\mathbb{Q}} \longrightarrow R_{\mathbb{Q}/S}$$

Consider

$$\begin{tikzcd} & \pi_{4k} S_{\mathbb{Q}/S} \arrow[r] \arrow[d, "\cong"] & \pi_{4k} R_{\mathbb{Q}/S} \ni b_{2k}^{\text{Miller}} \\ \pi_{4k-1}(O) \arrow[r, "J"] \arrow[d, "\cong"] & \pi_{4k-1} S & \\ \pi_{4k}(KO) \arrow[d] & & \\ \pi_{4k} KU \ni u^{2k} \arrow[uuurr, bend right, leftrightarrow] & & \end{tikzcd}$$

Thm $(\{b_k\}, \mu) \in [C_{g_N}, (gl_1 R)_{E(1)}]$ corresponds to an orientation $\iff$

$$b_{2k} \equiv b_{2k}^{\text{Miller}} \in \pi_{4k} R_{\mathbb{Q}/S_p}$$

IV Resumé on Ando – Hopkins – Smith

e.g. $\hat{A}$ genus $\qquad b_k^{\text{Miller}} = \dfrac{-B_k}{2k}$

$$\mu = \text{Mazur measure}$$

$$\int_{(\mathbb{Z}_p^\times)/(\pm 1)} x^{2k}\, d\mu = (\ell^{k}-1)(1-p^{k-1})\left(-\frac{B_{2k}}{4k}\right)$$

$$= \frac{1}{2}\, \zeta^{*,\ell}(1-2k)$$

p-adic ζ-function

[**Note:** (on factor of 2)

$$\mathbb{Z}_p^\times \approx \mathbb{Z}_p^\times/_{\pm 1} \amalg \mathbb{Z}_p^\times/_{\pm 1}$$

μ extends to a measure on $\mathbb{Z}_p^\times$

$$\int_{\mathbb{Z}_p^\times} x^k\, d\mu = (\ell^k - 1)(1-p^{k-1})\, \zeta^{*,\ell}(1-k)$$
]

Witten genus $\qquad$ $\underline{Q\text{-expansion}}$

$$b_{2k}^{\text{Miller}}(q) = \frac{-B_{2k}}{4k} \in \mathbb{Q}/_{\mathbb{Z}}[[q]]$$

Rmk: This can be argued as follows:

[Miller invariant is natural in E_∞-ring spectra.]

$$\begin{tikzcd} tmf \arrow[dr, "q\text{-expansion}"] & & \\ & KO[[q]] \simeq KO \wedge \mathbb{N}_+ & \\ K \arrow[ur, hook] & & \end{tikzcd}$$

map of E_∞-ring spectra

$\mathbb{V}_{tmf}$ = Katz p-adic modular functions

$\pi_{2k} tmf_{K(1)}$ = p-adic modular fns of wt k

μ = Katz measure valued in $\mathbb{V}_{tmf}$

$$\int_{\mathbb{Z}_p^\times} x^k \, d\mu = (\ell^k - 1)\, 2G_k^*$$

$$G_k^* = \log_1 G_k = G(z) - p^{k-1} G(pz)$$

And of course

$$G_k(q) \equiv -\frac{B_k}{2k} \mod \mathbb{Z}[[q]]$$

At this point we have an orientation

$$MString \xrightarrow[w]{} tmf_{E(1)}$$

$$K_w(x) = \exp\left(\sum_{k \geq 2} 2G_k \frac{x^k}{k!}\right)$$

Want to lift this to an $E(2)$-local orientation

Use chromatic fracture:

$$
\begin{tikzcd}[column sep=small]
 & (gl_1\, tmf)_{E(2)} \arrow[dl, "\log_0"'] \arrow[d, "\log_1"] \arrow[dr, "\log_2"] & \\
tmf_{\mathbb{Q}} \ni \{b_n\} \arrow[d, "(\log_1)_{\mathbb{Q}}"'] \arrow[dr, "(\log_2)_{\mathbb{Q}}"] & tmf_{K(1)} \ni \mu \arrow[dl] \arrow[dr, "\log_2'"] & tmf_{K(2)} \ni f_{K(2)} \text{ (see below)} \arrow[dl] \arrow[d] \\
tmf_{K(1),\mathbb{Q}} \arrow[dr, "(\log_2')_{\mathbb{Q}}"'] & tmf_{K(2),\mathbb{Q}} \arrow[d] & tmf_{K(2),K(1)} \arrow[dl] \\
 & tmf_{K(2),K(1),\mathbb{Q}} &
\end{tikzcd}
$$

and

$$\log_1 = 1 - \frac{1}{p} V_p$$

$$\log_2 = 1 - T_p + \frac{1}{p} R_p$$

$$\log_2' = 1 - U_p$$

$$
\begin{tikzcd}
(b\mathrm{String})_{K(2)} \arrow[r] \arrow[d, phantom, "\wr\!\!\downarrow"] & (gl_1 S)_{K(2)} \arrow[r, "\simeq", "\gamma"'] \arrow[d] & (C_{\mathrm{String}})_{K(2)} \arrow[dl, dashed, "f_{K(2)}"] \\
* & (gl_1 \mathrm{tmf})_{K(2)} &
\end{tikzcd}
$$

$\exists !$ factorization

<u>Compatibility</u> of $f_{K(2)}$ with μ

<u>∞-loop Adams conj</u>:

$$
\begin{tikzcd}
bu_p \arrow[r, "\psi^\ell - 1"'] \arrow[rr, bend right, "*"'] & bu_p \arrow[r] & bgl_1 S_p
\end{tikzcd}
$$

$\Longrightarrow$

$$
\begin{tikzcd}
bu_p \arrow[r] \arrow[d, dashed, "\alpha_u"'] & bu_p \arrow[d] \\
(C_u)_p \arrow[r] & (bgl_1 S)_p
\end{tikzcd}
$$

<u>Compatibility w/ α</u>:

$$
\begin{tikzcd}
bu\langle N\rangle_p \arrow[r] \arrow[d, "\alpha_u"] & (bo\langle N\rangle)_{K(1)} \arrow[r, phantom, "\simeq"] & KO_p \arrow[d, "\alpha"', "\simeq"] \\
(C_u)_p \arrow[r] & (C_{g_N})_{K(1)} \arrow[r, phantom, "\simeq"] & C_{g_N}
\end{tikzcd}
$$

So

$$
\begin{tikzcd}
\delta_{K(2)} \arrow[d, phantom, "\cap" rotate=90] & & & 0 \arrow[d, phantom, "="] \\
{[C_{g_N}, \mathrm{tmf}_{K(2)}]} \arrow[r] \arrow[d] & {[C_u, -]} \arrow[rr, "\alpha_u^*"] & & {[bu\langle N\rangle, \mathrm{tmf}_{K(2)}]} \arrow[d] \\
{[C_{g_N}, \mathrm{tmf}_{K(2),K(1)}]} \arrow[r, "\alpha^*"'] & {[KO_p, \mathrm{tmf}_{K(2),K(1)}]} \arrow[rr, hook] & & {[KU_p, \mathrm{tmf}_{K(2),K(1)}]} \\
& & & {[KU_p, (gl_1 \mathrm{tmf})_{K(1)}]} \arrow[u, "1-U_p"'] \arrow[d, phantom, "\ni" rotate=-90] \\
& \text{the induced measure on } \mathbb{Z}_p^\times \arrow[rr] & & \mu
\end{tikzcd}
$$

We showed:

Thm:

$\delta_{K(2)}$ is compatible with μ iff $(1-U_p)(\mu) = 0$

i.e. If $m_k \in \int_{\mathbb{Z}_p^\times} x^k \, d\mu$

need $(1-U_p)(m_k) = 0$

compatibility of $\delta_{K(2)}$ w/ $\{b_k\}$:

$$\begin{tikzcd}
\delta_{K(2)} \\
{[gl_1 S, tmf_{K(2)}]} \arrow[u, phantom, "\in" description] & {[C_{S_{\mathbb{Q}}}, tmf_{K(2)}]} \arrow[l, "\cong"'] \arrow[d] \\
{[C_{u_N}, tmf_{K(2),\mathbb{Q}}]} & {[C_{S_{\mathbb{Q}}}, tmf_{K(2),\mathbb{Q}}]} \arrow[l, hook'] \\
& {[C_{S_{\mathbb{Q}}}, tmf_{\mathbb{Q}}]} \arrow[u, "\log_2"] & {[bo\langle N\rangle, tmf_{\mathbb{Q}}]} \arrow[l, "\beta", "\cong"'] \\
& & \{b_K\} \arrow[u, phantom, "\in" description]
\end{tikzcd}$$

defined on homotopy gps

so suffices to consider complex analog

$$\begin{tikzcd}
gl_1 S \arrow[r, "\gamma"'] & C_{g_N} \arrow[r, "\beta"'] & bo\langle N\rangle \\
gl_1 S \arrow[u, no head] \arrow[r, "\gamma_u"'] & C_{u_N} \arrow[u] \arrow[r, "\beta_u"'] & bu\langle N\rangle \arrow[u] \arrow[r] & bgl_1 S \\
& & bu\langle N\rangle \arrow[ul, dashed, "\alpha_u"] \arrow[u, "\psi^\ell - 1"'] \arrow[ur, "*"']
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \pi_{2k}\, gl_1 S \arrow[d, "\gamma_u"] \arrow[dr] & \\
 & \pi_{2k}\, C_u \arrow[r, "\delta_{K(2)}"'] \arrow[d, "\beta_u"] & \pi_{2k}\, tmf_{K(2)} \arrow[d] \\
\pi_{2k}\, bu\langle N\rangle \arrow[ur, "\alpha"] \arrow[r, "\psi^\ell - 1"'] & \pi_{2k}\, bu\langle N\rangle \arrow[r, "h"'] & \pi_{2k}\, tmf_{K(2),\mathbb{Q}}
\end{tikzcd}
$$

Let $x \in (\pi_{2k} C_u)_{\mathbb{Q}}$

Suppose $(h\ \beta_u)_{\mathbb{Q}}(x) \neq 0$

$$(\beta_u)_{\mathbb{Q}} x = (\psi^\ell - 1)\frac{(\beta_u)_{\mathbb{Q}}(x)}{(\ell^k - 1)}$$

$$= (\beta_u\ \alpha)_{\mathbb{Q}} \underbrace{\frac{(\beta_u)_{\mathbb{Q}}(x)}{(\ell^k - 1)}}_{y}$$

So $(h\beta_u)_{\mathbb{Q}}\, x = (h\beta_u\alpha)_{\mathbb{Q}}(y)$

$$= (\delta_{K(2)})_{\mathbb{Q}}(\alpha)_{\mathbb{Q}}(y)$$

But $\delta_{K(2)}\alpha \simeq *$ since $[bu\langle N\rangle, tmf_{K(2)}] = 0$

We have shown:

Lem: $(\mathfrak{f}_{K(2)})_{\mathbb{Q}} \simeq *$

Thm $\mathfrak{f}_{K(2)}$ is compatible w/ $\{b_k\}$

iff $\log_2 b_k = 0$

$$\left[\text{i.e.} \quad (1 - T_p + p^{k-1}) b_k = 0\right]$$

Now: $m_k = (\ell^k - 1) t_k^*$

where $t_k^* = \log_1 (t_k)$

$$(1 - U_p) m_k = 0 \iff (1 - U_p) t_k^* = 0$$

$$(1 - U_p)(t_k^*) = (1 - U_p)(1 - \tfrac{1}{p} V_p)(t_k)$$

$$= (1 - T_p + p^{k-1})(t_k)$$

Thus $\mathfrak{f}_{K(2)}$, μ, $\{b_k\}$ are compatible

iff $\int_{\mathbb{Z}_p^\times} x^k \, d\mu = 2 b_k$, $(1 - T_p + p^{k-1})(b_k) = 0$

$(\mu, \{b_k\})$ gives an orientation if

$$b_k \equiv b_k^{\text{Miller}} \in \pi_4\, \mathrm{tmf}_{\mathbb{Q}_p/S_p}$$

$b_k = G_k$, $\mu = \text{Katz}$ measure does the trick!

get

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_{\text{String}} \arrow[dl, dashed, "(\{G_k\},\, \mu_{\text{Katz}})"] \\ (gl_1 \mathrm{tmf})_{E(2)} & \end{tikzcd}$$

Note this is same as

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & C_{\text{String}} \arrow[dl, dashed] \\ gl_1 \mathrm{tmf} & \end{tikzcd}$$

$$\underset{\substack{\parallel \\ [\text{String},\, d_2\mathrm{tmf}]}}{0} \longrightarrow \underset{\substack{\cap \\ [\text{String},\, gl_1 \mathrm{tmf}]}}{ob_{\text{String}}} \longrightarrow [\text{String},\, (gl_1\mathrm{tmf})_{K(2)}]$$

$$\pi_{>3}(d_2 \mathrm{tmf}) = 0$$

$$\pi_{<7}\, \text{String} = 0$$

V Eisenstien series on Shimura varieties

V.1 holomorphic automorphic forms on GL_2

$$K_\infty \subset GL_2(\mathbb{R})^+$$
$$\| $$
$$\left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| \ x+iy \in \mathbb{C}^\times \right\}$$
$$\cong ?$$
$$GL_1(\mathbb{C})$$

$GL_2(\mathbb{R}) \circlearrowright \mathcal{H} = \mathbb{C} - \mathbb{R} =$ double half plane.

$K_\infty =$ stabilizer of i , $\mathcal{H} \approx GL_2(\mathbb{R})/K_\infty$

$$\chi_k : K_\infty \longrightarrow \mathbb{C}^\times$$
$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \longmapsto (x+iy)^{-k}$$

irred rep of K_∞

$$\mathcal{V}_k \cong GL_2(\mathbb{R}) \times_{K_\infty} \mathbb{C}^{\chi_k}$$
$$\downarrow$$
$$\mathcal{H} \cong GL_2(\mathbb{R})/K_\infty$$

This holomorphic line bundle is trivialized w/ a section $\sigma_k : [g] \longmapsto \mu(g,i)^{-k}$

$$\mu(g,i) := ci+d$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$k_\infty = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in K_\infty$

[**Now** $\quad \mu(g k_\infty, i) = \mu(k_\infty, i)\,\mu(g, k_\infty i) = \chi_k(k_\infty)\,\mu(g, i)$]

The section σ_k determines a factor of automorphy:

$$\mu_k : GL_2(\mathbb{R}) \times \mathcal{H} \longrightarrow \mathbb{C}^\times$$

$$\mu_k(gg', \tau) = \mu_k(g, g'\tau)\,\mu_k(g', \tau)$$

Via $\quad \tau = g' i$

$$\mu_k(g, \tau) = \mu_k(g, g'i) = \frac{\mu_k(gg', i)}{\mu_k(g', i)} = \frac{\sigma_k(gg')}{\sigma_k(g')}$$

$$GL_2(\mathbb{A}^\infty) \times \mathcal{H} \cong GL_2(\mathbb{A}) / K_\infty$$

"Adelic analog of $\mathcal{H}$"

Extend σ_k to $GL_2(\mathbb{A}) \cong GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$

$$\sigma_k(g) = \sigma_k(g_\infty)$$

Similarly extend μ_k to $GL_2(\mathbb{A}) \times GL_2(\mathbb{A}^\infty) \times \mathcal{H}$

$\underline{\text{Via}}$ $\quad \mu_k(g, (h^\infty, h_\infty i)) = \dfrac{\sigma_k(g_\infty h_\infty)}{\sigma_k(h_\infty)}$

$$= \mu_k(g_\infty, h_\infty i)$$

<u>holomorphic Automorphic forms:</u>

$$\mathcal{A}_k^{hol} = \left\{ \phi : GL_2(\mathbb{A}) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ \phi(\gamma g) = \phi(g) \\ \quad \forall \gamma \in GL_2(\mathbb{Q}) \\ (2)\ \phi(g k_\infty) = \chi_k(k_\infty)\phi(g) \\ \quad k_\infty \in K_\infty \\ (3)\ g_\infty \mapsto \phi(g^\infty g_\infty) \\ \quad \text{is a hol fcn of } \mathcal{D}_k/\mathcal{H} \\ (4)\ \phi \text{ is smooth on} \\ \quad GL_2(\mathbb{A}^\infty) \times \{g_\infty\} \\ (5)\ \phi \text{ satisfies a} \\ \quad \text{growth cndtn} \end{array} \right\}$$

Holomorphic automorphic forms of wt k

$$GL_2(\mathbb{A}^\infty) \circlearrowright \mathcal{A}_k^{hol}$$

$$(4) \Rightarrow \quad \mathcal{A}_k^{hol} = \varinjlim_{\substack{K^\infty \subset GL_2(\mathbb{A}^\infty) \\ \text{cpt open}}} \left(\mathcal{A}_k^{hol}\right)^{K^\infty}$$

Automorphic forms $\rightsquigarrow$ modular forms

$$K^\infty \underset{\substack{\text{cpt}\\\text{open}}}{\subset} GL_2(\mathbb{A}^\infty)$$

$$\Gamma := GL_2(\mathbb{Q})^+ \cap K^\infty$$

$$\phi \in \left(\mathcal{A}_k^{\text{hol}}\right)^{K^\infty}$$

<u>Defn</u> $f : \mathcal{H} \longrightarrow \mathbb{C}$

$$f(\tau) = \mu_k(g_\infty, i)^{-1} \phi(1, g_\infty)$$

$\tau = g_\infty i$

here we use $GL_2(\mathbb{A}) \cong GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$

[Independent of $[g_\infty] \in GL_2(\mathbb{R})/K_\infty$

$k_\infty \in K_\infty$

$$\mu_k(g_\infty k_\infty, i)^{-1} \phi(1, g_\infty k_\infty) = \mu_k(g_\infty, i)^{-1} \phi(1, g_\infty)$$]

$\gamma \in \Gamma$

$$f(\gamma\tau) = \mu_k(\gamma g_\infty, i)^{-1} \phi(1, \gamma g_\infty)$$

$$= \mu_k(\gamma, \tau)^{-1} \mu_k(g_\infty, i)\, \underbrace{\phi(\gamma, \gamma g_\infty)}_{\phi(1, g_\infty)}$$

$$= \mu_k(\gamma, \tau)^{-1} f(\tau)$$

(5) growth condition $\iff$ f holomorphic at cusps

V.2: Moduli of elliptic curves

$V = \mathbb{R}^2$

$\Lambda = \mathbb{Z}^2$

$K^\infty \subset GL_2(\mathbb{A}^\infty)$ cpt open

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K^\infty K_\infty \underset{(*)}{\cong} \{(C, [\eta]_{K^\infty})\} / \sim$$

$C =$ elliptic curve $/ \mathbb{C}$

$\eta : V^\infty \xrightarrow[\cong]{} V(C)$

$\sim \; = \;$ isogeny

We regard $\quad GL_2(\mathbb{R}) / K_\infty = \left\{ \begin{matrix} \mathbb{C}\text{-structures on} \\ V \end{matrix} \right\}$

Define V_J to be corresponding $\mathbb{C}$-v.s.

$C_J = V_J / \Lambda$

given $g^\infty \in GL_2(\mathbb{A}^\infty)$

$$\eta_{g^\infty} : V^\infty \xrightarrow[g^\infty]{} V^\infty \xrightarrow[\text{can}]{\cong} V(C_J)$$

The iso (*) above is given by

$$GL_2(\mathbb{Q})\backslash\left(GL_2(\mathbb{A}^\infty)/K^\infty \times GL_2(\mathbb{R})/K_\infty\right)$$

$$\Psi$$

$$[g^\infty, J] \longmapsto (C_J, [\eta_{g^\infty}])$$

There is an induced line bundle

$$\mathcal{V}_k(K^\infty)$$

$$\downarrow$$

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}^\infty)/K^\infty K_\infty = X_{K^\infty}$$

$$\left(\mathcal{A}_k^{hol}\right)^{K^\infty} \cong \{\text{hol sections on } \overline{X}_{K^\infty}\}$$

V.3: Eisenstein series on GL_2

$\omega, \omega' : \mathbb{Q}^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ Hecke characters

$P \subseteq GL_2$ parabolic

$\{\begin{pmatrix} a & x \\ & b \end{pmatrix}\}$ get rep: $\rho(\omega, \omega') : P(\mathbb{A}) \longrightarrow \mathbb{C}^\times$

$$\begin{pmatrix} a & x \\ & b \end{pmatrix} \longmapsto \omega(a)\omega'(b)$$

$$\mathbb{A}^\times \cong \mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+^\times$$

$$(aa^\infty, aa_\infty) \longleftarrow (a, a^\infty, a_\infty)$$

ω^∞ - dirichlet character $\hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$

$$\omega \longleftrightarrow (\omega^\infty, \omega_\infty)$$

Assume: $\omega_\infty(x) = x^{-k_1}$ $\qquad k = k_1 + k_2 \in \mathbb{Z}$

$\omega'_\infty(x) = x^{-k_2}$

Induced rep of $GL_2(\mathbb{A}^\infty)$

$$\pi(\omega, \omega') = \left\{ f : GL_2(\mathbb{A}) \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ f \text{ is smooth on } GL_2(\mathbb{A}^\infty) \\ (2)\ f(g k_\infty) = \chi_k(k_\infty) f(g) \\ (3)\ \text{The induced section of } \mathcal{V}_k \text{ is holomorphic} \\ (4)\ f\left(\begin{pmatrix} a & * \\ & b \end{pmatrix} g\right) = \omega(a)\omega'(b) f(g)^{*} \end{array} \right\}$$

Eisenstein series;

embed $\pi(\omega, \omega')$ in space of automorphic forms:

Fix f

Defn $\qquad E_f : GL_2(\mathbb{A}) \longrightarrow \mathbb{C}$

$$E_f(g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash GL_2(\mathbb{Q})} f(\gamma g)$$

* it is traditional to normalize this with a 'modular character' - but I have not done so here.

E_f is clearly a holomorphic automorphic form provided the series converges, and the growth conditions are met.

Idea! if $k_2 - k_1 > 2$ integers

then E_f converges

$$\Longrightarrow \quad \pi(\omega, \omega') \xhookrightarrow{GL_2(\mathbb{A}^\infty)} \mathcal{A}_k^{hol}$$

$$f \longmapsto E_f$$

---

e.g. Define $\omega_k(\alpha a^\infty, \alpha a_\infty) = a_\infty^{-k}$

$(aa^\infty, aa_\infty) \in \mathbb{A}^\times$ $\quad$ $\alpha \in \mathbb{Q}$, $a^\infty \in \hat{\mathbb{Z}}^\times$, $a_\infty \in \mathbb{R}_+^\times$ $\quad$ so $\omega_k(a) = |a|^{-k}$

Lem 1 There is a bijection

$$(g^\infty, g_\infty) \longmapsto g^\infty g_\infty$$

$$U(\hat{\mathbb{Z}}) \times U(\mathbb{R}) \backslash GL_2(\hat{\mathbb{Z}}) \times X \xrightarrow{\cong} P(\mathbb{Q})\, U(\mathbb{A}) \backslash GL_2(\mathbb{A})$$

where $X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \;\middle|\; \begin{array}{l} \det > 0 \\ d > 0 \text{ or } d = 0,\ c > 0 \end{array} \right\}$

and $$P(\mathbb{Z})^+ \backslash SL_2(\mathbb{Z}) \xrightarrow{\cong} P(\mathbb{Q}) \backslash GL_2(\mathbb{Q})$$

$U = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \subset GL_2$

unipotent subgp $\qquad$ $(c,d) = 1$

[Assume k even]

[Note: $f \in \pi(\omega, \omega')$ is left $P(\mathbb{Q})U(\mathbb{A})$-invariant]

Define $f_k(g^\infty, g_\infty) = \dfrac{1}{(c_\infty i + d_\infty)^k} = \mu_k(g_\infty, i)$

$g^\infty \in GL_2(\hat{\mathbb{Z}})$

$\begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix} = g_\infty \in X$

Now if $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \in P(\mathbb{A})$

$$\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y_1 a + xc & y_1 b + xd \\ y_2 c & y_2 d \end{pmatrix}$$

RHS lies in $GL_2(\hat{\mathbb{Z}}) \times X$ if

(1) $y_1^\infty, y_2^\infty \in \hat{\mathbb{Z}}^\times$ $\quad x^\infty \in \hat{\mathbb{Z}}$

(2) $(y_1)_\infty, (y_2)_\infty > 0$

Given such $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}$, we check

$$f_k\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g\right) = \frac{1}{((y_2)_\infty c_\infty i + (y_2)_\infty d_\infty)^k} = (y_2)_\infty^{-k} \frac{1}{(c_\infty i + d_\infty)^k}$$

$$= \omega_k(y_2)\, f_k(g)$$

thus $f_k \in \pi(1, \omega_k)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + dw \\ cx + dz & cy + dw \end{pmatrix}$$

$$E_{f_k}(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash GL_2(\mathbb{Q})}' f_k(\gamma g)$$

$$= \sum_{\gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z})}' M_k(\gamma g_\infty, i)$$

$$\Updownarrow$$

$$\underset{\tau = g_\infty' i}{E_k(\tau)} = M_k(g_\infty', i)^{-1} \sum_{\gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z})}' M_k(\gamma g_\infty, i)$$

$$= \sum_{\gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z})}' M_k(\gamma, g_\infty i) = \sum_{\gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z})}' M_k(\gamma, \tau)$$

$$= \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c,d \in \mathbb{Z}}} \frac{1}{(c\tau + d)^k}$$

comes from $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in P(\mathbb{Z})^+$

More generally: $\qquad \chi : (\mathbb{Z}/N)^\times \longrightarrow \mathbb{C}^\times$

Suppose: χ is a dirichlet character (mod N)

$\rightsquigarrow$ Hecke character $\quad \omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$

$$\omega(g^\infty, g_\infty) = \chi(g^\infty)^{-1} g_\infty^{-k}$$

$$g^\infty \in \hat{\mathbb{Z}}^\times, \quad g_\infty \in \mathbb{R}_+^\times$$

Define $\quad f_k^{\chi} : \; GL_2(\hat{\mathbb{Z}}) \times X \big/ U(\hat{\mathbb{Z}}) \times U(\mathbb{R}) \longrightarrow \mathbb{C}$

$$\cong$$

$$GL_2(\mathbb{A}) \big/ P(\mathbb{Q}) U(\mathbb{A})$$

by:

$$f_k^{\chi}(g^{\infty}, g_{\infty}) = \begin{cases} \dfrac{1}{(c_{\infty} i + d_{\infty})^k} \chi^{-1}(d^{\infty}), & c^{\infty} \equiv 0 \ (N) \\ 0, & c^{\infty} \not\equiv 0 \ (N) \end{cases}$$

$$(g^{\infty}, g_{\infty}) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$g_{\infty} \in X$

$g^{\infty} \in GL_2(\hat{\mathbb{Z}})$

$$\left[\text{Assume:} \quad \chi(-1) = (-1)^k \right]$$

Then if $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \in P(\mathbb{A})$

$y_1^{\infty}, y_2^{\infty} \in \hat{\mathbb{Z}}^{\times}$, $x \in \hat{\mathbb{Z}}$

$(y_1)_{\infty}, (y_2)_{\infty} \in \mathbb{R}_+^{\times}$

$$\Longrightarrow \quad f_k^{\chi}\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g \right) = \omega(y_2) f_k^{\chi}(g)$$

$g = g^{\infty} g_{\infty}$

$g_{\infty} \in X$

$g^{\infty} \in GL_2(\hat{\mathbb{Z}})$

So $\quad f_k^{\chi} \in \pi(1, \omega)$

$$E_{f_k^\chi}(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash GL_2(\mathbb{Q})} f_k^\chi(\gamma g)$$

$g = g^\infty g_\infty$

$g^\infty \in GL_2(\hat{\mathbb{Z}})$

$g_\infty \in X$

$g^\infty = \begin{pmatrix} * & * \\ u & v \end{pmatrix}$

$$= \begin{cases} 0, & u \not\equiv 0 \quad (N) \\ \displaystyle\sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z}) \\ c \equiv 0 \ (N)}} M_k(\gamma g_\infty, i)\, \chi^{-1}(dv) \end{cases}$$

$\Updownarrow$

$$E_k^\chi(\tau) = M_k(g_\infty', i)^{-1} \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z}) \\ c \equiv 0 \ (N)}} M_k(\gamma g_\infty, i)\, \chi^{-1}(d)$$

$\tau = g_\infty' i$

$$= \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in P(\mathbb{Z})^+\backslash SL_2(\mathbb{Z}) \\ c \equiv 0 \ (N)}} M_k(\gamma, g_\infty i)\, \chi^{-1}(d) = \sum_{\substack{\gamma \in P(\mathbb{Z})^+\backslash \Gamma_0(N) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} M_k(\gamma, \tau)\, \chi^{-1}(d)$$

$$\frac{1}{2} \sum_{\substack{(c,d) = 1 \\ c, d \in \mathbb{Z} \\ c \equiv 0 \ (N)}} \frac{1}{(c\tau + d)^k}\, \chi^{-1}(d)$$

comes from $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in P(\mathbb{Z})^+$

V.4 Unitary Shimura varieties

F
$|$ quad imag ex
$\mathbb{Q}$

$\delta \in F \qquad \bar{\delta} = -\delta$

$F \hookrightarrow \mathbb{C}$

$\delta \longmapsto$ negative multiple of i

$V =$ d-dim'l F v.s. $\beta \in M_n(F)$ $\beta^* = -\beta$

$\langle \, , \rangle_\beta$ = $\mathbb{Q}$-valued alternating form

$\langle x, y \rangle_\beta = \mathrm{Tr}_{F/\mathbb{Q}}(x \beta y^*)$

$$\beta = \begin{bmatrix} & & -1_m \\ & -i\theta & \\ 1_m & & \end{bmatrix} \quad \text{in } M_n(\mathbb{C}) \quad \theta > 0$$

$V \cong V_0 \oplus V_1 \oplus \overline{V_0}$

V_1 = anisotropic $d - 2m$ dim'l

V_0 = totally isotropic m dim'l

$GU_{V/\mathbb{Q}}$ = unitary similitude gp $U_{V/\mathbb{Q}}$ = unitary gp

3 cases of interest:

(1) $m = 1$

(2) $d = 2m$

(3) $m = 0$

similitude norm

$$U_V \to GU_V \xrightarrow[\nu]{} \mathbb{G}_m$$

$GU(\mathbb{R})^+ = \{ g \mid \nu(g) \in \mathbb{R}_+^\times \}$

Z_∞ = center $\cong \mathbb{C}^\times$

$K_\infty \subset U(\mathbb{R})$ maximal cpt

$K_\infty \cong U(d-m) \times U(m)$

$GU(\mathbb{R}) / K_\infty Z_\infty = \mathcal{H} = \mathcal{H}^+ \amalg \mathcal{H}^-$

$\mathcal{H}^+ \cong GU(\mathbb{R})^+ / K_\infty Z_\infty \cong U(\mathbb{R}) / K_\infty$

2 descriptions of $\mathcal{H}^+$

(i) explicit:

$$\mathcal{H}^+ \cong \left\{ z = \begin{array}{cc} m & \\ \begin{bmatrix} x \\ y \end{bmatrix} & \begin{array}{l} m \\ d-2m \end{array} \end{array} \;\middle|\; \begin{array}{l} x \in M_m(\mathbb{C}) \\ y \in M_{d-2m,m}(\mathbb{C}) \\ i(x^* - x) > y^* \theta^{-1} y \end{array} \right\}$$

(ii) Abstract: space of compatible cx structures:

$$\mathcal{H}^+ \cong \left\{ J \in \mathrm{Aut}_F(V_\infty) \;\middle|\; J^2 = -1,\ \langle -, J- \rangle \ \begin{array}{l}\text{symmetric}\\ \text{+ positive}\end{array} \right\}$$

$GU(\mathbb{R})^+ \curvearrowright \mathcal{H}^+$ w/ stabilizer $Z_\infty K_\infty$

$g: J \longmapsto gJg^{-1}$

$K^\infty \underset{\substack{\text{cpt}\\ \text{opn}}}{\subset} GU(\mathbb{A}^\infty)$

$$Sh_{(GU, K^\infty)}(\mathbb{C}) = GU(\mathbb{Q})^+ \backslash \left(GU(\mathbb{A}^\infty)/K_\infty \times \mathcal{H}^+ \right)$$

$$\cong GU(\mathbb{Q}) \backslash \left(GU(\mathbb{A}^\infty)/K_\infty \times \mathcal{H} \right)$$

$$\cong GU(\mathbb{Q}) \backslash GU(\mathbb{A}) / K^\infty K_\infty Z_\infty$$

$L \subset V$ $\quad \mathcal{O}_F$-lattice

assume $\langle L, L \rangle \subseteq \mathbb{Z}$

$$K_0^\infty = \{ g \in GU(\mathbb{A}^\infty) \mid g(\hat{L}) = \hat{L} \}$$

$$Sh(\mathbb{C}) \cong \{(A, i, \lambda, [\eta]_{K_0^\infty})\}/\text{isogeny}$$

A = d-dim'l ab var. $/\mathbb{C}$

$i : F \hookrightarrow \mathrm{End}^\circ(A)$ cx mult

λ : polarization

$\eta : V^\infty \xrightarrow[\cong]{} V(A)$ $\dim \mathrm{Lie}\, A^+ = m$

$\langle\, ,\rangle \xmapsto[\text{similitude}]{} \langle\, ,\rangle_\lambda$ $\dim \mathrm{Lie}\, A^- = d-m$

$J \in \mathcal{H}^+$

$W_J = V \otimes \mathbb{R}$ w/ cx structure J

$A_J = W_J/L$

i_J = cx mult from F v.s structure on $W = V_\infty$

λ_J = polarization associated to

$(-,-)_J = \langle J-, -\rangle + i\langle -,-\rangle$ Riemann Form

$\eta_1 : V^\infty \longrightarrow V(A)$ canonical level structure

$\eta_g : V^\infty \xrightarrow{g} V^\infty \xrightarrow{\eta_1} V(A)$ $g \in GU(\mathbb{A}^\infty)$

$$GU(\mathbb{Q}^+)\backslash \left(GU(\mathbb{A}^\infty)/K_0^\infty \times \mathcal{H}\right) \xrightarrow{\cong} Sh(\mathbb{C})$$

$$[g^\infty, J] \longmapsto (A_J, i_J, \lambda_J, [\eta_J g^\infty]_{K_0^\infty})$$

Note: in case (3): $m = 0$

$$\mathcal{H}^+ = *$$

$$\Rightarrow \quad Sh(\mathbb{C}) \cong GU(\mathbb{Q}) \backslash GU(\mathbb{A}^\infty) / K_0^\infty \qquad \text{discrete space}$$

Canonical point:

$$(A_J, i_J, \lambda_J, [\eta_1]) \in Sh(\mathbb{C})$$

In general

given $K_\infty \xrightarrow[\rho]{} \mathbb{C}^n$ irrep

ρ extends to Z_∞

get $\mathcal{V}_\rho^{K^\infty} = GU(\mathbb{Q})^+ \backslash \left(GU(\mathbb{A}^\infty)/K^\infty \times GU(\mathbb{R})^+ \times_{K_\infty Z_\infty} \mathbb{C}^n \right)$

$$\downarrow$$

$$Sh(\mathbb{C}) = GU(\mathbb{Q})^+ \backslash \left(GU(\mathbb{A}^\infty)/K^\infty \times \mathcal{H}^+ \right)$$

$$\mathcal{A}_\rho^{hol}(GU) = \left\{ \phi : GU(\mathbb{A}) \longrightarrow \mathbb{C}^n \;\middle|\; \begin{array}{l} (1)\ \phi(\gamma g) = \phi(g) \\ \quad \forall\, \gamma \in GU(\mathbb{Q})^+ \\ (2)\ \phi(g k_\infty) = \rho(k_\infty)\phi(g) \\ \quad k_\infty \in K_\infty Z_\infty \\ (3)\ g_\infty \mapsto \phi(g^\infty g_\infty) \\ \quad \text{is a hol action of } \mathcal{D}_\rho / \mathcal{K}^+ \\ (4)\ \phi \text{ is smooth on } \\ \quad GU(\mathbb{A}^\infty) \times \{g_\infty\} \\ (5)\ \phi \text{ satisfies a} \\ \quad \text{growth condition} \end{array} \right\}$$

Holomorphic automorphic forms of wt ρ

$$GU(\mathbb{A}^\infty) \circlearrowright \mathcal{A}_\rho^{hol}(GU)$$

$$\mathcal{A}_\rho^{hol}(GU)^{K^\infty} = \Gamma_{hol}\left(\begin{array}{c} V_\rho(K^\infty) \\ \downarrow \\ Sh(K^\infty) \end{array} \;\middle|\; \begin{array}{l} \text{condition (5)} \\ \Rightarrow \text{extends over} \\ \text{compactification} \end{array} \right)$$

For us important examples of ρ are:

(1) $m = 1$,

$$\chi_k : K_\infty = U(d-1) \times U(1) \longrightarrow U(1) \xrightarrow{z \mapsto z^{-k}} \mathbb{C}^\times$$

(2) $d = 2m$,

$$\chi_k' : K_\infty = U(m) \times U(m) \xrightarrow{\pi_2} U(m) \xrightarrow{\det^{-k}} \mathbb{C}^\times$$

V.5 Eisenstein series

Parabolic induction

$$P_{\bar V_0} = \text{maximal parabolic} \subset GU_V$$

$$= \{ g \mid g(\bar V_0) = \bar V_0 \} \qquad GL_{V_0}(\mathbb{Q}) \cong GL_m(F)$$

If we regard elts of V as Row vectors, and elts of GU as matrices acting on V on right

$$V = V_0 \oplus V_1 \oplus \bar V_0$$

Then

$$P_{\bar V_0} = \left\{ \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix} \right\} \subset GU_V$$

Consider

$$P_{\bar V_0}(\mathbb{A}) \longrightarrow GL_{\bar V_0} \xrightarrow[\omega(\det)]{} \mathbb{C}^\times$$

Assume

$$\begin{tikzcd} \mathbb{A}_F^\times \cong (\mathbb{A}_F^\infty)^\times \times \mathbb{C}^\times \arrow[r, "\omega"'] & \mathbb{C}^\times \\ \mathbb{C}^\times \arrow[u, hook] \arrow[ur, "\chi_K"'] & \end{tikzcd}$$

Need to assume: $\chi_K \big|_{M_F} = \text{trivial}$

We get an unnormalized induced rep:

$$\pi(1,\omega) = \left\{ f : GU(\mathbb{A}) \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ f \text{ is smooth on } GU(\mathbb{A}^\infty) \\ (2)\ f(g k_\infty) = \chi_K(\det(k_\infty)) f(g) \\ (3)\ \text{The induced section of } \chi_K \text{ is holomorphic} \\ (4)\ f\left(\begin{pmatrix} * & * & * \\ * & * & * \\ & & a \end{pmatrix} g \right) = \omega(\det(a)) f(g) \end{array} \right\}$$

Eisenstein series

$k > 2m + 2(d-2m) = 2d - 2m$

$$\pi(1,\omega) \hookrightarrow \mathcal{A}_k^{hol}(GU_V)$$
$$f \longmapsto E_f$$

$$E_f(g) = \sum_{\gamma \in P_{\bar{V}_0}(\mathbb{Q}) \backslash GU_V(\mathbb{Q})} f(\gamma g)$$

converges for k above

Pullback of Eisenstein Series:

Our set-up:

$$V = V_0 \oplus V_1 \oplus \overline{V_0}$$
$$W = V_0 \oplus V_1 \oplus \overline{V_0} \oplus \overline{V_1}$$
$$= W_0 \oplus \overline{W_0}$$

$\overline{V_1} = -V_1$

$\dim V_1 = n-2$

$\dim V_0 = 1$

$\dim V = n$

$\dim W = 2n-2$

$\dim W_0 = n-1$

$W_0 = \{(v_0, v_1, 0, v_1)\}$

$\overline{W_0} = \{(0, v_1, \bar{v}_0, -v_1)\}$

Naturally, we have

$$GU_V \hookrightarrow GU_W$$
$$g \longmapsto g \oplus 1_{-V_1}$$

Fix a Hecke character ω as before

Embedding of Symmetric hermitian domains:

$$\mathcal{H}_V^+ \cong \left\{ J_V \in \mathrm{Aut}_F(V_\infty) \,\middle|\, J_V^2 = -1,\ \langle -, J_V - \rangle_V \ \begin{matrix}\text{symmetric}\\ \text{+ positive}\end{matrix} \right\}$$

$$\mathcal{H}_W^+ \cong \left\{ J_W \in \mathrm{Aut}_F(W_\infty) \,\middle|\, J_W^2 = -1,\ \langle -, J_W - \rangle_W \ \begin{matrix}\text{symmetric}\\ \text{+ positive}\end{matrix} \right\}$$

Recall $\mathcal{H}_{V_1}^+ \cong \mathcal{H}_{-V_1}^+ = *$

$\Longrightarrow \exists!\ J_{V_1} \in \mathrm{Aut}_F(V_1)$, s.t. $J_{V_1}^2 = -1$

$\langle -, J - \rangle_{V_1}$ symmetric + positive.

Note: under $V \oplus -V_1 \cong W$

we have $\langle (x,y), (x',y') \rangle_W = \langle x, x' \rangle_V - \langle y, y' \rangle_{V_1}$

Consider the map:

$$\begin{array}{ccc} \mathcal{H}_V^+ \times \mathcal{H}_{V_1}^+ & \xrightarrow[\equiv]{\hspace{2cm}} & \mathcal{H}_W^+ \\ \text{\rotatebox{90}{$\cong$}} & & \rotatebox{90}{$\in$} \\ \mathcal{H}_V^+ \times \{*\} & & \\ \rotatebox{90}{$\in$} & & \\ J_V & \longmapsto & J_V \oplus -J_{V_1} \end{array}$$

here

$$J_V \oplus -J_{V_1} : \quad \begin{matrix} W & \longrightarrow & W \\ \shortparallel & & \shortparallel \\ V \oplus -V_1 & & V \oplus -V_1 \\ (x, y) & \longmapsto & (J_V(x), -J_{V_1}(y)) \end{matrix}$$

Note that

$$\langle x, J_V x' \rangle_V + \langle y, J_{V_1} y' \rangle_{V_1} \qquad \text{is symmetric \& positive.}$$

<u>Equivariance</u>

$$g \in GU_V(\mathbb{R})$$

$$g_1 \in GU_{-V_1}(\mathbb{R})$$

get $\quad g \oplus g_1 \in GU_W(\mathbb{R})$

note: $g_1 J_{V_1} g_1^{-1} = J_{V_1}$ by uniqueness

$$(g \oplus g_1) J_V \oplus -J_{V_1} (g \oplus g_1)^{-1} = g J_V g^{-1} \oplus - g_1 J_{V_1} g_1^{-1}$$

$$\Longrightarrow \quad \Xi \text{ is } GU_V^+(\mathbb{R}) \times GU_{-V_1}^+(\mathbb{R}) \text{ - equivariant}$$

Fix a basepoint $\underline{J}_V \in \mathcal{H}_V^+$

Let $K_\infty \subset U_V(\mathbb{R})$ be the stabilizer of $\underline{J}_V$

$K_\infty' \subset U_W(\mathbb{R})$ is the stabilizer of $\underline{J}_V \oplus -J_{V_1}$

$$\begin{tikzcd} GU_V^+(\mathbb{R}) \arrow[r, hook] & GU_W^+(\mathbb{R}) \\ K_\infty Z_\infty \arrow[u, hook] \arrow[r] & K_\infty' Z_\infty \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} GU_V^+(\mathbb{R})/K_\infty Z_\infty \arrow[r, hook] & GU/K_\infty' Z_\infty \\ \mathcal{H}_V^+ \arrow[u, phantom, "\cong"] \arrow[r, "\Xi"] & \mathcal{H}_W^+ \arrow[u, phantom, "\cong"] \end{tikzcd}$$

Embedding of Shimura Varieties

Fix $\mathcal{O}_F$-lattices

$$L_0 \subset V_0 \quad (\Rightarrow \text{ set } \bar{L}_0 \subset \bar{V}_0)$$

$$L_1 \subset V_1$$

get $$L = L_0 \oplus L_1 \oplus \bar{L}_0 \subset V \rightsquigarrow K_L^\infty \subset GU_V(\mathbb{A}^\infty)$$

$$L' = L_0 \oplus L_1 \oplus \bar{L}_0 \oplus \bar{L}_1 \subset W \rightsquigarrow K_{L'}^\infty \subset GU_W(\mathbb{A}^\infty)$$

Get:

$$GU_V(\mathbb{Q})^+ \backslash \left(GU_V(\mathbb{A}^\infty)/K_L^\infty \times \mathcal{H}_V^+ \right) \longrightarrow GU_W(\mathbb{Q})^+ \backslash \left(GU_W(\mathbb{A}^\infty)/K_{L'}^\infty \times \mathcal{H}_W^+ \right)$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$Sh_V(\mathbb{C}) \qquad\qquad\qquad\qquad Sh_W(\mathbb{C})$$

$$[g^\infty, J_V] \qquad\qquad\qquad\qquad\qquad \Downarrow$$

$$\updownarrow$$

$$\underline{A}_{J_V, g^\infty} \longmapsto \underline{A}_{J_V, g^\infty} \oplus \underline{A}_{-J_V, 1}$$

$$\|$$

$$(A_{J_V}, i_{J_V}, \lambda_{J_V}, [\eta_{J_V} g^\infty]_{K_L^\infty})$$

Pullback of automorphic V.B.'s

$$\begin{tikzcd}
K_\infty \cong U(n-1)\times U(1) \arrow[r] \arrow[d] \arrow[rr, bend left, "x_\kappa"] & U(1) \arrow[r, "z \mapsto z^{-\kappa}"'] & \mathbb{C}^\times \arrow[d, equal] \\
K'_\infty \cong U(n-1)\times U(n-1) \arrow[r] \arrow[rr, bend right, "x'_\kappa"'] & U(n-1) \arrow[r, "\det^{-\kappa}"'] & \mathbb{C}^\times
\end{tikzcd}$$

Thus: get pullback

$$\begin{tikzcd}
\mathcal{V}_{x_\kappa} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \mathcal{V}_{x'_\kappa} \arrow[d] \\
Sh_V(\mathbb{C}) \arrow[r, "\Xi"'] & Sh_W(\mathbb{C})
\end{tikzcd}$$

$\Rightarrow$ get: $\Xi^* : \mathcal{A}^{hol}_{x'_\kappa}(GU_W)^{K_L^\infty} \longrightarrow \mathcal{A}^{hol}_{x_\kappa}(GU_V)^{K_L^\infty}$

pullback of automorphic forms

$$\phi \longmapsto \left(g \mapsto \phi\begin{pmatrix} g & \\ & 1_{-v_1}\end{pmatrix}\right)$$

Let ω be the Hecke Character

$$\omega : (\mathbb{A}_F^\infty)^\times \longrightarrow \mathbb{C}^\times$$

So that $\omega\big|_{F_\infty^\times} : F_\infty^\times = \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ $\quad g^\infty \in \hat{\mathbb{Z}}^\times$

$z \longmapsto z^{-k}$ $\quad g_\infty \in \mathbb{R}_+^\times$

Assume $|\mu_F| \mid k$

Get an induced map of parabolically induced Reps:

$$\pi_W(1,\omega) \xrightarrow[\Xi^*]{} \pi_V(1,\omega)$$

$$f \longmapsto \left(g \longmapsto f\left(\begin{smallmatrix} g & \\ & 1_{V_1} \end{smallmatrix}\right)\right)$$

Finally, we are ready to pullback Eisenstein Series.

<u>Thm</u> Suppose $k > 2n-2$. Then:

$$\Xi^* E_f = E_{\Xi^* f}$$

and both converge.

(pf) Note that if $k > 4(n-1) - 2(n-1)$

$$\Longrightarrow k \geq 2n-2$$

So we are in the range of convergence for Eisenstein series for both GU_V and GU_W

<u>lemma:</u> for both V and W:

$$P(\mathbb{Q}) \backslash GU(\mathbb{Q}) \cong P'(\mathbb{Q}) \backslash U(\mathbb{Q})$$

where: $P'(\mathbb{Q}) = P(\mathbb{Q}) \cap U(\mathbb{Q})$

<u>pf</u> of lemma: $\nu : GU(\mathbb{Q}) \longrightarrow \mathbb{Q}^\times$, $GU \leftrightarrow V_0 \oplus V_1 + \overline{V_0}$ in our previous notation

It suffices to show

$$\operatorname{Im} \nu|_{P(\mathbb{Q})} \supseteq \operatorname{Im} \nu$$

But I think! Im ν only depends on d mod 2

$$\begin{cases} N(F^\times) & d \equiv 1 \ (2) \\ \mathbb{Q}^\times & d \equiv 0 \ (2) \end{cases}$$

and $\nu|_{P(\mathbb{Q})} \supseteq \operatorname{Im}(\nu : GU_{V_1} \longrightarrow \mathbb{Q}^\times)$

$$\shortparallel$$

$$\operatorname{Im}(\nu : GU_V \longrightarrow \mathbb{Q}^\times)$$

and $\dim V_1 \equiv \dim V \mod 2$.

$\overline{V}_0 \hookrightarrow \overline{W}_0$

$\Rightarrow \quad P_{\overline{V}_0}(\mathbb{Q}) \hookrightarrow P_{\overline{W}_0}(\mathbb{Q})$ | $P_{\overline{V}_0}(\mathbb{Q}) = P_{\overline{W}_0(\mathbb{Q})} \cap GU_V$

Lemma (Shimura: Euler products & Eisenstein Series, Prop 2.4)

$$P'_{\overline{V}V_0}(\mathbb{Q}) \backslash U_W(\mathbb{Q}) / U_V(\mathbb{Q}) \times U_{-V_1}(\mathbb{Q}) = *$$

Then we compute: $g \in GU_V(\mathbb{A})$

$$E_f\begin{pmatrix} g & \\ & 1 \end{pmatrix} = \sum_{\gamma \in P_{\overline{W}_0}(\mathbb{Q}) \backslash GU_W(\mathbb{Q})} f\left(\gamma \begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$$

$$= \sum_{\gamma \in P'_{\overline{W}_0}(\mathbb{Q}) \backslash U_W(\mathbb{Q})}' f\left(\gamma \begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$$

$$= \sum_{\begin{pmatrix} \gamma_1 & \\ & \gamma_2 \end{pmatrix} \in P'_{\overline{W}_0}(\mathbb{Q}) \cap (U_V \times U_{-V_1}) \backslash (U_V(\mathbb{Q}) \times U_{-V_1}(\mathbb{Q}))}' f\left(\begin{pmatrix} \gamma_1 & \\ & \gamma_2 \end{pmatrix} \begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$$

Note: given any $\gamma_2 \in U_{V_1}$, $\begin{pmatrix} 1_{\gamma_2} & \\ & 1 & \\ & & \gamma_2 \end{pmatrix} \in P'_{\overline{W}_0}(\mathbb{Q})$

$\Rightarrow$ can assume $\gamma_2 = 1$

$$\Longrightarrow \quad = \sum_{\gamma_1 \in P'(\mathbb{Q}) \backslash U(\mathbb{Q})} f\left({}^{\gamma_1 g} \, , \right)$$

$$= \sum_{\gamma \in P(\mathbb{Q}) \backslash \{U(\mathbb{Q})} f\left({}^{\gamma g} \, , \right) = E_{\Xi^* f}(g)$$

$\square$