SHIMURA VARIETIES OF TYPE $U(1, n-1)$ IN CHARACTERISTIC ZERO

The complex theory of the moduli space of elliptic curves, and its associated elliptic modular forms, admits an elementary exposition involving the upper half plane. These notes are supposed give a similarly tractable description of the Shimura varieties of type $U(1, n-1)$, and the holomorphic automorphic forms associated to these, following:

- R.E. Kottwitz, *Points on some Shimura varieties over finite fields.*
- M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties.*
- H. Hida, *p-Adic automorphic forms on Shimura varieties.*
- A. Borel, *Introduction to Automorphic forms.*

It has become common practice to treat the moduli problems represented by these Shimura varieties as certain *isogeny classes* of weakly polarized abelian varieties with complex multiplication and level structure. The isogeny class approach, (due to Deligne?) gives the cleanest formulation of the moduli. There are many subtleties associated to this moduli problem which it hides, however, so in this document we shall formulate the moduli in terms of *isomorphism classes*. These subtleties are then more plainly exposed.

1. Notation

Let $\mathbb{A}$ denote the rational adeles. If S is a set of places of $\mathbb{Q}$, we shall use the convention that places in the superscript are omitted and places in the subscript are included.

$$\mathbb{A}_S = \prod_{v \in S}{}' \mathbb{Q}_v$$

$$\mathbb{A}^S = \prod_{v \notin S}{}' \mathbb{Q}_v$$

This notational philosophy will be extended to other contexts as we see fit.

Begin with the following data:

$$\begin{aligned}
F &= \text{quadratic imaginary extension of } \mathbb{Q}. \\
\mathcal{O}_F &= \text{ring of integers.} \\
B &= \text{central simple algebra over } F,\ \dim_F B = n^2. \\
(-)^* &= \text{positive involution on } B \text{ of the second kind.} \\
\mathcal{O}_B &= \text{maximal order in } B. \\
V &= \text{left } B\text{-module.} \\
(-,-) &= \mathbb{Q}\text{-valued non-degenerate alternating form on } V \text{ which is } *\text{-hermitian.} \\
& \qquad \text{(This means } (\alpha x, y) = (x, \alpha^* y).\text{)} \\
L' &= \mathcal{O}_B\text{-invariant lattice in } V,\ (-,-) \text{ restricts to give integer values on } L'.
\end{aligned}$$

We will sometimes let the choice of L' vary — this will correspond to looking at various connected components of the Shimura variety. We fix a particular lattice L to act as a "basepoint". From this data we define:

$$\begin{aligned}
\widehat{L}' &= \text{the profinite completion of } L' \\
C &= \mathrm{End}_B(V). \\
(-)^\iota &= \text{involution on } C \text{ defined by } (av, w) = (v, a^\iota w). \\
\mathcal{O}_C(L') &= \text{order of elements } x \in C \text{ such that } x(L') \subseteq L'. \\
G(R) &= \{g \in (C \otimes_{\mathbb{Q}} R)^\times \ : \ g^\iota g \in R^\times\} \\
U(R) &= \{g \in (C \otimes_{\mathbb{Q}} R)^\times \ : \ g^\iota g = 1\} \\
K^\infty &= \{g \in G(\mathbb{A}^\infty) \ : \ g(\widehat{L}) = \widehat{L}\} \\
G(\mathbb{Q})^+ &= \{g \in G(\mathbb{Q}) \ : \ g^\iota g > 0\} \\
\Gamma(L') &= \{g \in G(\mathbb{Q})^+ \ : \ g(L') = L'\}
\end{aligned}$$

We are interested in the case where we have:

$$\begin{aligned}
V &= B \\
L &= \mathcal{O}_B \\
U(\mathbb{R}) &= U(1, n-1)
\end{aligned}$$

It then follows that we have

$$\begin{aligned}
C &= B^{op} \\
\mathcal{O}_C(L) &= \mathcal{O}_B^{op} \qquad \text{(by maximality of } \mathcal{O}_B) \\
\Gamma(L') &= \{g \in \mathcal{O}_C(L')^\times \ : \ g^\iota g \in \mathbb{Q}_{>0}\}.
\end{aligned}$$

In our case $*$ may equally well be regarded as an involution $*$ on C, and there exists (using the Noether-Skolem theorem) an element $\beta \in B$ so that $\beta^* = -\beta$ that encodes $(-,-)$:

$$(x, y) = \mathrm{Tr}_{F/\mathbb{Q}} \mathrm{Tr}_{B/F}(x\beta y^*).$$

Let γ be the element β regarded as an element of C. Then ι is given by

$$z^\iota = \gamma^{-1} z^* \gamma.$$

Tensoring with $\mathbb{R}$, and fixing a complex embedding of F, we may identify the completions of our simple algebras with matrix algebras over $\mathbb{C}$:

$$\begin{aligned}
B_\infty &= M_n(\mathbb{C}) \\
* &= \text{conjugate transpose} \\
C_\infty &= M_n(\mathbb{C}) \quad \text{(identified with } B \text{ through the transpose)} \\
\beta &= \begin{bmatrix} e_1 i & & & \\ & -e_2 i & & \\ & & \ddots & \\ & & & -e_n i \end{bmatrix}
\end{aligned}$$

Here the e_i's are positive real numbers.

Let $\epsilon \in B_\infty$ be the idempotent consisting of a 1 in the $(1,1)$ entry and zeros elsewhere. The summand ϵV_∞ is then the subspace of matrices in B_∞ with non-zero entries concentrated in the first row.

$$\epsilon V_\infty = \left\{ \begin{bmatrix} v_1 & \cdots & v_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} : v_i \in \mathbb{C} \right\} \subset B_\infty$$

We may thus view elements of ϵV_∞ as vectors $\vec{v} = (v_1, \dots, v_n)$ in $\mathbb{C}^n$ Under this identification the algebra $C_\infty = M_n(\mathbb{C})$ acts on ϵV_∞ by the standard representation.

For $x = [x_{i,j}] \in C_\infty$, the involution ι is given by:

$$x^\iota = \begin{bmatrix} \overline{x}_{1,1} & -\frac{e_2}{e_1}\overline{x}_{2,1} & \cdots & -\frac{e_n}{e_1}\overline{x}_{n,1} \\ -\frac{e_1}{e_2}\overline{x}_{1,2} & \overline{x}_{2,2} & \cdots & \frac{e_n}{e_2}\overline{x}_{n,2} \\ \vdots & \vdots & & \vdots \\ -\frac{e_1}{e_n}\overline{x}_{1,n} & \frac{e_2}{e_n}\overline{x}_{2,n} & \cdots & \overline{x}_{n,n} \end{bmatrix}$$

The pairing $(-,-)$ restricts to an $\mathbb{R}$-valued alternating pairing on ϵ, given explicitly by the formula

$$(\vec{v}, \vec{w}) = \mathrm{Im}(-2e_1 v_1 \overline{w}_1 + 2e_2 v_2 \overline{w}_2 + \cdots + 2e_n v_n \overline{w}_n).$$

There is a natural way to associate an i-hermitian form $H_i(-,-)$ to $(-,-)$, by the formula

$$H_i(\vec{v}, \vec{w}) = (i\vec{v}, \vec{w}) + i(\vec{v}, \vec{w}).$$

(The subscript i on $H_i(-,-)$ indicates the dependence this construction on the complex structure i on V_∞.) The pairing $(-,-)$ is then recovered by $(-,-) = \mathrm{Im}\, H_i(-,-)$. The pairing $H_i(-,-)$ is given by

$$H_i(\vec{v}, \vec{w}) = -2e_1 v_1 \overline{w}_1 + 2e_2 v_2 \overline{w}_2 + \cdots + 2e_n v_n \overline{w}_n.$$

Let $x \in C_\infty$ be a matrix with columns $\vec{c}_i \in \mathbb{C}^n$. Then we have:

$$x \in G(\mathbb{R}) \quad \text{if there is an } r \in \mathbb{R}^\times \text{ so that } \begin{cases} H_i(\vec{c}_1, \vec{c}_1) = -2re_1, & \\ H_i(\vec{c}_j, \vec{c}_j) = 2re_j & \text{for } j > 1, \\ H_i(\vec{c}_i, \vec{c}_j) = 0 & \text{for } i \neq j. \end{cases} \tag{1.1}$$

$$x \in U(\mathbb{R}) \quad \text{if we have } \begin{cases} H_i(\vec{c}_1, \vec{c}_1) = -2e_1, & \\ H_i(\vec{c}_j, \vec{c}_j) = 2e_j & \text{for } j > 1, \\ H_i(\vec{c}_i, \vec{c}_j) = 0 & \text{for } i \neq j. \end{cases} \tag{1.2}$$

2. The hermitian symmetric domain

Say $J \in C_\infty$ is a *complex structure* if it satisfies:

(1) $J^2 = -1$.
(2) $J^\iota = -J$.

Because $i \in C_\infty$ is central, if J is a complex structure, then the product iJ satisfies $(iJ)^2 = 1$, and we may therefore decompose any hermitian C_∞-module W as $W_J^+ \oplus W_J^-$ where

$$W_J^+ = W^{i=J}$$
$$W_J^- = W^{i=-J}.$$

We shall say that a complex structure J is *compatible* if:

(1) $\dim_{\mathbb{C}}(V_\infty)^+_J = n$, or equivalently, $\dim_{\mathbb{C}}(\epsilon V_\infty)^+_J = 1$.
(2) The pairing (v, Jw) is symmetric on V_∞.
(3) The pairing (v, Jw) is positive on V_∞.

These three conditions equivalent to:

(1) J has 1-dimensional i-eigenspace and $(n-1)$-dimensional $-i$-eigenspace.
(2) J commutes with γ.
(3) $\gamma = \rho J$ for ρ a positive matrix which commutes with γ and J (the eigenvalues of ρ are necessarily $\{e_1, ..., e_n\}$).

Let $\mathcal{H}$ be the space of compatible complex structures. An example of a compatible complex structure is given by

$$I = \begin{bmatrix} i & & & \\ & -i & & \\ & & \ddots & \\ & & & -i \end{bmatrix} \in C_\infty.$$

We then have

$$\begin{aligned}(\epsilon V_\infty)^+_I &= \{(v_1, 0 \dots 0)\}\\ (\epsilon V_\infty)^-_I &= \{(0, v_2, \dots v_n)\}.\end{aligned}$$

Proposition 2.1. The group $U(\mathbb{R})$ acts transitively on the space $\mathcal{H}$ by conjugation.

Proof. Because γ, ρ, and J all commute, they are simultaneously diagonalizable. The positivity of ρ, together with the dimensionality of the $\pm i$-eigenspaces of J show that there exists a unitary x so that

$$\begin{aligned}\gamma &= x\gamma x^*\\ I &= xJx^{-1}.\end{aligned}$$

We see that x lies in $U(\mathbb{R})$. Thus every compatible complex structure lies in the $U(\mathbb{R})$-orbit of I. □

Let K_∞ be the stabilizer of $I \in \mathcal{H}$.

Proposition 2.2. The subgroup K_∞ is given by

$$U((\epsilon V_\infty)^+_I) \times U((\epsilon V_\infty)^-_I).$$

The group K_∞ is therefore abstractly isomorphic to $U(1) \times U(n-1)$.

Proof. Suppose that x is an element of $U(\mathbb{R})$ so that

$$xIx^{-1} = I.$$

Then we deduce

$$\begin{aligned}1 &= I^{-1}x^{-1}Ix\\ &= I^{-1}x^\iota Ix\\ &= I^{-1}\gamma^{-1}x^*\gamma Ix.\end{aligned}$$

Now we have

$$\gamma I = \begin{bmatrix} -e_1 & & \\ & \ddots & \\ & & -e_n \end{bmatrix}.$$

Let $H_I(-,-)$ be the I-hermitian form on ϵV_∞ given by

$$\begin{aligned} H_I(\vec{v}, \vec{w}) &= (I\vec{v}, \vec{w}) + i(\vec{v}, \vec{w}) \\ &= -2e_1 v_1 \overline{w}_1 - \cdots - 2e_n v_n \overline{w}_n. \end{aligned}$$

We deduce that if x has column vectors $\vec{c}_i$, then we must have

$$x \in K_\infty \quad \text{if and only if } \begin{cases} H_I(\vec{c}_j, \vec{c}_j) = -2e_j, & \\ H_I(\vec{c}_i, \vec{c}_j) = 0 & \text{for } i \neq j. \end{cases}$$

It is therefore clear that we have the containment

$$U((\epsilon V_\infty)_I^+) \times U((\epsilon V_\infty)_I^-) \subseteq K_\infty.$$

Writing x as $[x_{i,j}]$, and comparing with condition (1.2), we see that

$$\begin{cases} e_1 |x_{1,1}|^2 = e_1 \\ e_1 |x_{1,1}|^2 + \cdots + e_n |x_{n,1}|^2 = e_1 \end{cases}$$

from which we deduce (since the e_i are positive) that $x_{j,1} = 0$ for $j \geq 1$ and so $\vec{c}_1$ is of the form $(x_{1,1}, 0, \ldots 0)$ for $|x_{1,1}|^2 = 1$. Since for $j \geq 1$ we have

$$0 = H_I(\vec{c}_j, \vec{c}_1) = -2e_1 x_{1,j} \overline{x}_{1,1},$$

we conclude that $x_{1,j} = 0$ for all $j > 0$. Therefore our element x takes the form

$$\left[\begin{array}{c|ccc} * & & & \\ \hline & * & \cdots & * \\ & \vdots & & \vdots \\ & * & \cdots & * \end{array}\right].$$

□

It turns out that the domain $\mathcal{H} = U(\mathbb{R})/K_\infty$ admits a canonical complex structure, but we do not pursue the details here.

3. Polarized abelian varieties over $\mathbb{C}$

A polarized abelian variety of dimension g over $\mathbb{C}$ is given by a lattice Λ in a g-dimensional complex vector space W together with a Riemann form $(-,-)$ on W. To be a Riemann form, the form $(-,-)$ is required to satisfy

(1) $(-,-)$ is an $\mathbb{R}$-valued $\mathbb{R}$-bilinear alternating form.
(2) $(-,-)$ takes integer values on Λ.
(3) $(-, i-)$ is symmetric and positive.

To such forms we may associate a hermitian form $H(-,-)$ via the formula

$$H(v, w) = (iv, w) + i(v, w).$$

Then $(-,-)$ is recovered as $\operatorname{Im} H(-,-)$. Define

$$\begin{aligned} W^* &= \{\alpha : W \to \mathbb{C} \, : \, \alpha \text{ conjugate linear}\} \\ \Lambda^* &= \{\alpha \in W^* \, : \, \operatorname{Im} \alpha(\Lambda) \subseteq \mathbb{Z}\}. \end{aligned}$$

Then Λ^* is a lattice in W^*. Associated to a non-degenerate Riemann form $(-,-)$ is a $\mathbb{C}$-linear isomorphism

$$\begin{aligned}\widetilde{\lambda} : W &\to W^* \\ v &\mapsto H(v,-).\end{aligned}$$

The isomorphism $\widetilde{\lambda}$ restricts to give a finite index embedding

$$\widetilde{\lambda} : \Lambda \hookrightarrow \Lambda^*.$$

Say the index is N. Then we may define a Riemann form $(-,-)_\lambda$ on W^* by the equation

$$(\alpha, \beta)_\lambda = N \cdot (\widetilde{\lambda}^{-1}(\alpha), \widetilde{\lambda}^{-1}(\beta)).$$

The torus $A = W/\Lambda$ is complex analytic. Given a non-degenerate Riemann form, we may produce an ample line bundle $\mathcal{L}$ on A. Therefore, A admits an embedding into projective space, giving A the structure of a projective variety (GAGA). The line bundle $\mathcal{L}$ gives rise to a polarization. The dual abelian variety is given by $A^\vee = W^*/\Lambda^*$, and the map $\widetilde{\lambda}$ descends to give the isogeny corresponding to the polarization of A induced by the Riemann form $(-,-)$:

$$\lambda : A \to A^\vee.$$

This completes a sketch of the equivalence of categories

$$\begin{gathered}\{\text{Lattices with non-degenerate Riemann forms}\} \\ \updownarrow \\ \{\text{Polarized abelian varieties}/\mathbb{C}\}\end{gathered}$$

4. The Tate module and the Weil pairing

For N a positive integer there is a Weil pairing e on $A[N]$ and $A^\vee[N]$ given by

(4.1)
$$\begin{tikzcd} A[N] \times A^\vee[N] \arrow[r, "e_N(-,-)"] \arrow[d, equal] & \mu_N \\ N^{-1}\Lambda/\Lambda \times N^{-1}\Lambda^*/\Lambda^* \arrow[r, "ev_N"'] & N^{-1}\mathbb{Z}/\mathbb{Z} \arrow[u, "\cong"', "\exp(2\pi i -)"] \end{tikzcd}$$

Here the map ev_N is given by

$$ev_N(v, \alpha) = N \cdot \operatorname{Im} \alpha(v).$$

The composite

$$\lambda_N(-,-) : A[N] \times A[N] \xrightarrow{1\times\lambda} A[N] \times A^\vee[N] \xrightarrow{e_N} \mu_N$$

is the λ-Weil pairing. Everything is compatible, so we may take the inverse limit over N to get a pairing on the Tate module

$$\widehat{T}(A) \times \widehat{T}(A) \xrightarrow{\lambda(-,-)} \widehat{\mathbb{Z}}(1).$$

We may likewise take the inverse limits of the pairings

$$(-,-) : \Lambda/N \times \Lambda/N \to \mathbb{Z}/N$$

to get a pairing

$$\widehat{\Lambda} \times \widehat{\Lambda} \to \widehat{\mathbb{Z}}.$$

Proposition 4.1. Under the isomorphisms

$$N^{-1}\Lambda/\Lambda \xrightarrow[\cong]{\cdot N} \Lambda/N\Lambda$$
$$N^{-1}\mathbb{Z}/\mathbb{Z} \xrightarrow[\cong]{\cdot N} \mathbb{Z}/N$$

and the isomorphisms of Diagram (4.1), the λ-Weil pairing is sent to the Riemann form $(-,-)$. In particular, there are canonical isomorphisms making the following diagram commute.

$$\begin{tikzcd} \widehat{\Lambda} \times \widehat{\Lambda} \arrow[r, "{(-,-)}"] \arrow[d, "\cong"'] & \widehat{\mathbb{Z}} \arrow[d, "\cong"] \\ \widehat{T}(A) \times \widehat{T}(A) \arrow[r, "{\lambda(-,-)}"'] & \widehat{\mathbb{Z}}(1) \end{tikzcd}$$

5. The coarse moduli functor

Let K be a field containing F. We denote the canonical embedding of F in K by u_+, and its conjugate embedding by u_-:

$$u_\pm : F \hookrightarrow K.$$

Suppose that W is a K-vector space (so it is in particular an F-vector space) and that there is a ring homomorphism

$$j : B \to \mathrm{End}_K(W).$$

Then there is a decomposition

$$W = W_j^+ \oplus W_j^-$$

where

$$W_j^+ = \{x \in W \ : \ F \text{ acts through } j \text{ by } u_+\}$$
$$W_j^- = \{x \in M \ : \ F \text{ acts through } j \text{ by } u_-\}.$$

A polarization $\lambda : A \to A^\vee$ defines a λ-*Rosati involution* $*$ on $\mathrm{End}^0(A)$ by $f^* = \lambda^{-1} f^\vee \lambda$. If the polarization is principle, the Rosati involution leaves $\mathrm{End}(A)$ invariant.

Define a contravariant functor

$$\mathcal{X} : \mathrm{Fields}/F \to \mathrm{Sets}$$

by associating to K/F a set of isomorphism classes of tuples of data

$$\mathcal{X}(K) = \{(A, \lambda, j, \overline{\eta})\}/\cong$$

where:

$$\begin{array}{lll}
A & = & \text{abelian variety over } K \text{ of dimension } n^2. \\
\lambda : A \to A^\vee, & & \text{a polarization.} \\
j : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) & \text{so that} & \begin{cases} j \text{ is an inclusion of rings.} \\ j(b^*) = j(b)^*, \text{ in } \mathrm{End}^0(A). \\ \dim_K(\mathrm{Lie}(A)_j^+) = n. \end{cases} \\
\eta : \widehat{L} \xrightarrow{\cong} \widehat{T}(A) & \text{so that } \eta & \begin{cases} \text{is } \mathcal{O}_B\text{-linear.} \\ \text{is } Gal(\overline{K}/K)\text{-invariant.} \\ \text{sends } (-,-) \text{ to an } (\mathbb{A}^\infty)^\times\text{-multiple of} \\ \qquad \text{the } \lambda\text{-Weil pairing.} \end{cases}
\end{array}$$

We declare that two tuples (A, λ, j, η) and $(A', \lambda', j', \eta')$ are equivalent if there is an isomorphism of abelian varieties

$$\phi : A \xrightarrow{\cong} A'$$

so that:

$$\begin{aligned}
\lambda &= r\phi^\vee \lambda' \phi, && r \in \mathbb{Q}^\times \\
j'(b)\phi &= \phi j(b), && b \in \mathcal{O}_B \\
\eta' &= \phi_* \eta k && \text{for some } k \in K^\infty.
\end{aligned}$$

We shall refer to the K^∞ orbit of η as $\overline{\eta}$. Note that the action of K^∞ equates any two choices of η. Nevertheless, it is important that one choice of η *exists.* We could rephrase our level structure as saying that the elements of $\prod_p H^1(Q_p, G)$ given by $\lambda(-,-)$ and $(-,-)$ coincide, together with an integral condition on the Tate module itself. This is discussed in the next two sections.

6. Forms of $(-,-)$

Let K be any field of characteristic 0. Let $V_K = V \otimes K$ be the $B_K = B \otimes K$-module with $*$-hermitian non-degenerate K-bilinear alternating pairing

$$(-,-) : V_K \otimes_K V_K \to K$$

induced from the pairing $(-,-)$ on V. We shall regard a different pairing $(-,-)'$ as *similar* to $(-,-)$ if there is an element z of $(C \otimes K)^\times$ so that

$$(zx, zy) = r(x, y)$$

for $r \in K^\times$. We shall say they are *equivalent* if $r = 1$. In particular, the automorphisms of $(-,-)$ are given by the similitude group $G(K)$. A different pairing $(-,-)'$ is a *form of* $(-,-)$ if it is similar to $(-,-)$ over $\overline{K}$. The forms of $(-,-)$ are classified by the non-abelian Galois cohomology group $H^1(K, G)$.

Thus the orbit of level structures $\overline{\eta}$ associated to a point $(A, \lambda, j, \overline{\eta}) \in \mathcal{X}(K)$ imposes an additional condition on the polarized abelian variety (A, λ): for each finite prime p, we require (by choosing an $\mathcal{O}_B$-linear isomorphism $L_p \cong T_p(A)$) that the λ-Weil pairing on $V_p(A) = T_p(A) \otimes \mathbb{Q}$ is similar to the p-local pairing $(-,-)_p$ on V_p.

We now assume that K is either $\mathbb{Q}$, or $\mathbb{Q}_v$ for a place v of $\mathbb{Q}$ which is not split in F. Then a non-degenerate $*$-hermitian alternating form $(-,-)'$ must be given by

$$(x, y)' = \mathrm{Tr}_{F\otimes K/K} \mathrm{Tr}_{B\otimes K/F\otimes K}(x\beta' y^*)$$

where $\beta' \in (B\otimes K)^{*=-1}$. Write γ' for the corresponding element of $C_K = B^{op}\otimes K$. Then $(-,-)'$ is similar to $(-,-)$ if and only if there exists an element $z \in C_K^\times$ so that

$$\gamma = rz^*\gamma' z$$

for $r \in K^\times$.

Now specialize to the case of $K = \mathbb{R}$. Using the polar decomposition of a transformation $z \in (C_\mathbb{R})^\times = GL_n(\mathbb{C})$, we deduce that an $\mathbb{R}$-form $(-,-)'$ of $(-,-)$ is classified by $|\mathrm{sign}\,\beta'|$, the absolute value of the signature $\mathrm{sign}(a, b) = b-a$, where β' has a positive imaginary eigenvalues and b negative imaginary eigenvalues.

Given a form $(-,-)'$ of $(-,-)$ over $\mathbb{Q}$, such that $(-,-)'_v$ is similar to $(-,-)_v$ over $\mathbb{Q}_v$ for all places v of $\mathbb{Q}$, is it similar to $(-,-)$? The answer is *yes*, because of the following proposition.

Proposition 6.1 (Kottwitz, Hida)**.** The map

$$H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G)$$

is injective.

Kottwitz proves this for n even and showed that the kernel is finite for n odd. Hida observes (p319) that because the totally real number field contained in F is $\mathbb{Q}$, there is no kernel for n odd as well (in our case, his group GU is equal to our G).

7. G-lattice classes

The complex points of $\mathcal{X}$ arise as quotients $A = V_\infty/L'$ for certain integral lattices L'. The existence of the level structure $\overline{\eta}$ indicates that the lattice L' must satisfy certain local conditions as well. In this section we will formulate a notion of equivalence of such lattices.

Say that a lattice $L' \subset V_\infty$ is of *type* L if there exist for each prime p, elements $g_p \in G(\mathbb{Q}_p)$ such that

$$g_p(L'_p) = L_p.$$

We shall say that two lattices L', L'' of type L are equivalent if there exists an element $g \in G(\mathbb{Q})^+$ so that

$$g(L') = L''.$$

Let $\mathrm{Cl}_L(G)$ denote the set of equivalence classes of lattices of type L. It is given by the double coset space

$$\mathrm{Cl}_L(G) \cong G(\mathbb{Q})^+\backslash G(\mathbb{A}^\infty)/K^\infty.$$

For convenience, we shall assume that we have chosen a preferred representative L' of each lattice equivalence class $[L']$. Note that if L' is a lattice of type L, and N is a positive integer, the lattice NL' is an equivalent lattice of type L. We may therefore assume that each representative L' is chosen so that $(-,-)$ takes integer values on L'.

8. Complex points

Define the *Shimura variety* $Sh^0_{\mathbb{C}}$ to be the disjoint union of complex quotients

$$Sh^0_{\mathbb{C}} = \coprod_{L' \in \mathrm{Cl}_L(G)} \Gamma(L')\backslash\mathcal{H}.$$

I put the superscript 0 to indicate that the Shimura variety is a coarse moduli space.

Theorem 8.1. There is a canonical isomorphism

$$Sh^0_{\mathbb{C}} \xrightarrow{\cong} \mathcal{X}(\mathbb{C}).$$

To a lattice L' and a point $J \in \mathcal{H}$, we shall associate a tuple

$$(A_{(L',J)}, \lambda, j, \overline{\eta}) \in \mathcal{X}(\mathbb{C}).$$

Namely, give V_∞ the complex structure afforded by J, and let:

$A_{(L',J)} = V_\infty / L'$.

$\lambda =$ polarization given by the non-degenerate Riemann form $(-,-)$.

$j =$ the obvious action of $\mathcal{O}_B$ on $A_{(L',J)}$.

$\overline{\eta} =$ the canonical level structure.

Note that Proposition 4.1 gives a canonical isomorphism $\widehat{T}(A_{(L',J)}) \cong \widehat{L}'$, so a level structure exists because the lattice L' is of type L. Any two choices of level structure differ by an element of K^∞. Thus we have a map

$$\Phi' : \coprod_{L' \in \mathrm{Cl}_L(G)} \mathcal{H} \to \mathcal{X}(\mathbb{C}).$$

An element $g \in \Gamma(L')$, gives rise to a different complex structure $J' = gJg^{-1}$. The complex structure J' is compatible: the positivity of $g^\iota g$ allows us to deduce the positivity condition on J' with respect to $(-,-)$. The element g acts naturally on V through $\mathcal{O}_B$-linear maps, sending J to J', and since it preserves L' it induces a j-linear isomorphism

$$g_* : A_{(L',J)} \to A_{(L',J')}.$$

Because g lies in $G(\mathbb{Q})$, it is a similitude of $(-,-)$, and thus it preserves the weak polarization. Therefore the map Φ' descends to a map

$$\Phi : \coprod_{L' \in \mathrm{Cl}_L(G)} \Gamma(L')\backslash\mathcal{H} \to \mathcal{X}(\mathbb{C}).$$

Φ is injective. Suppose that (J', L') and (J'', L'') are two pairs of lattices and compatible complex structures that yield equivalent elements of $\mathcal{X}(\mathbb{C})$ under the map Φ. Let

$$\phi : (A_{(L',J')}, \lambda, j, \overline{\eta}) \to (A_{(L'',J'')}, \lambda, j, \overline{\eta})$$

be the isomorphism. Because ϕ must preserve the polarization, we have

$$\lambda = r\phi^\vee \lambda \phi.$$

Taking the degree of both sides, we see that $r = 1$. Now ϕ must arise from a map

$$g : V_\infty \to V_\infty$$

which maps the lattice L' onto L'' and such that $J''g = gJ'$. We want to show that g lies in $\Gamma(L')$. Because ϕ preserves the polarization on the nose, the map g

must preserve $(-,-)$ on the nose. Because it sends a rational lattice to a rational lattice, we deduce that g actually lies in $\mathrm{End}_{\mathbb{Q}}(V)$. We have deduced that g lies in $U(\mathbb{Q})$, so in particular we have $g^\iota g \in \mathbb{Q}_{>0}$. We deduce that $L' = L''$, because we have shown these lattices represent the same lattice class. Because ϕ is $\mathcal{O}_B$-linear, we see that g is $\mathcal{O}_B$-linear, and therefore (because it preserves the lattice) we have $g \in \mathcal{O}_C(L')^\times$. Therefore g lies in $\Gamma(L')$. We have therefore shown that the map Φ is injective.

Φ is surjective. The surjectivity of the map Φ is not typical of the PEL Shimura varieties of type A, as observed by Kottwitz. Rather, one expects

$$\mathcal{X}(\mathbb{C}) = \coprod_{i \in \ker^1(\mathbb{Q}, G)} \coprod_{L' \in \mathrm{Cl}_L(G^{(i)})} \Gamma(L') \backslash \mathcal{H}^{(i)}$$

where $\ker^1(\mathbb{Q}, G)$ is the finite kernel of the map

$$H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G).$$

However, in our case (Proposition 6.1), this kernel *is* trivial. We shall soon see how this sort of thing enters into the discussion.

We demonstrate surjectivity: Suppose that $(A', \lambda', j', \overline{\eta}')$ is a point in $\mathcal{X}(\mathbb{C})$. The pair (A', λ') gives a complex vector space W, a lattice $\Lambda \subset W$, and a non-degenerate Riemann form $(-,-)'$ on (W, Λ). The order $\mathcal{O}_B$ acts through j' on Λ by lattice automorphisms. Since B is simple, we may we once and for all identify

$$\Lambda \cong L'$$
$$W \cong V_\infty.$$

for some $\mathcal{O}_B$-lattice L' in V_∞ of type L. The pairing $(-,-)'$ induces a non-degenerate alternating pairing $(-,-)'$ on V_∞ which probably differs from $(-,-)$. The fact that j' sends $*$ to the the Rosati involution implies that $(-,-)'$ is $*$-hermitian. The complex structure on W induces a complex structure J' on V_∞ which is compatible with $(-,-)'$. Because $(-,-)'$ gives integer values on L', it actually lifts to a rational pairing $(-,-)'$ on V. Let λ' be the polarization of $A_{(L',J')}$ given by the form $(-,-)'$. Take $\overline{\eta}'$ to be the level structure on $A_{J'}$ induced from the j-linear isomorphism $A' \cong A_{(L',J')}$ of weakly polarized abelian varieties. We then get an isomorphism

$$(A', \lambda', j', \overline{\eta}') \cong (A_{(L',J')}, \lambda', j, \overline{\eta}').$$

We just need to show that we can change J' to a complex structure compatible with $(-,-)$, so that $(-,-)'$ and $\overline{\eta}'$ get changed to $(-,-)$ and $\overline{\eta}$.

Associated to the weird pairing $(-,-)'$:

- There exists $\beta' \in B^{*=-1}$ so that
$$(x, y)' = \mathrm{Tr}_{F/\mathbb{Q}} \mathrm{Tr}_{B/F}(x \beta' y^*).$$
- β' may be regarded as $\gamma' \in C$, and the involution ι' associated to $(-,-)'$ is given by
$$z^{\iota'} = (\gamma')^{-1} z^* \gamma'.$$

- Because $(V_\infty)^+_{J'}$ is 1-dimensional, J' diagonalizes to
$$J' \sim \begin{bmatrix} i & & & \\ & -i & & \\ & & \ddots & \\ & & & -i \end{bmatrix}.$$
- Because $(-, J'-)'$ is symmetric, we may deduce that J' commutes with γ', and so they simultaneously diagonalize, giving
$$\gamma' \sim \begin{bmatrix} e'_1 i & & & \\ & -e'_2 i & & \\ & & \ddots & \\ & & & -e'_n i \end{bmatrix}.$$
- Because $(-, J'-)'$ is positive, we may deduce that the entries e'_i in the matrix above are positive real numbers.

We conclude that the forms $(-,-)'$ and $(-,-)$ are similar over $\mathbb{R}$, because they have the same signature. The level structure guarantees that the forms $(-,-)'$ and $(-,-)$ are similar over $\mathbb{Q}_p$ for all p. Thus, by Proposition 6.1 the forms $(-,-)'$ and $(-,-)$ are similar over $\mathbb{Q}$. Therefore, there exists an element $x \in C^\times$ giving an $\mathcal{O}_B$-linear similitude
$$x : (V, (-,-)') \xrightarrow{\sim} (V, (-,-)).$$

The similitude x has a problem: it need not be positive, and therefore the complex structure $x^{-1}J'x$ need not be compatible with $(-,-)$. However, there is a decomposition
$$G(\mathbb{R}) = G(\mathbb{R})^+ G(\mathbb{Q}). \tag{8.1}$$
At the infinite place we used the fact that $|\mathrm{sign}(\gamma')| = |\mathrm{sign}(\gamma)|$, but we showed that there was an equality of the signatures without taking absolute values. Therefore there exists an element $y_\infty \in G(\mathbb{R})^+$ giving a positive similitude (similitude norm is positive)
$$y_\infty : (V_\infty, (-,-)') \to (V_\infty, (-,-)).$$
Let $z \in G(\mathbb{R})$ be a similitude which completes the diagram
$$\begin{tikzcd} & (V_\infty, (-,-)) \arrow[d, "z"] \arrow[dr, "z_1"] & \\ (V_\infty, (-,-)') \arrow[ur, "x"] \arrow[r, "y_\infty"'] & (V_\infty, (-,-)) \arrow[r, "z_\infty^{-1}"'] & (V_\infty, (-,-)) \end{tikzcd}$$
where $z = z_\infty z_1$ is a decomposition by (8.1), for $z_1 \in G(\mathbb{Q})$ and $z_\infty \in G(\mathbb{R})^+$. Let $w = z_1 x$ be the element in $G(\mathbb{Q})$ which, by the commutativity of the above diagram, lies in $G(\mathbb{R})^+$. We deduce that
$$w : (V, (-,-)') \to (V, (-,-))$$
is a positive similitude. Let L'' be the lattice
$$L'' = w(L').$$

Because L' is of type L, the lattice L'' is also of type L. By altering w by an element of $G(\mathbb{Q})^+$, we may assume that L'' is one of our preferred representatives in $\mathrm{Cl}_L(G)$. The transformation w gives an isomorphism

$$w : A_{(L',J')} \xrightarrow{\cong} A_{(L'',J'')}$$

where $J = w^{-1}J'w$. It is easy to check that J is compatible with $(-,-)$. Therefore J lies in the hermitian symmetric space $\mathcal{H}$. Giving the abelian variety $A_{(L'',J)}$ the polarization λ associated to $(-,-)$, and a level structure $\overline{\eta}$ arising from the fact that L'' is of type L, the isomorphism w induces an isomorphism

$$w : (A_{(L',J')}, \lambda', j, \overline{\eta}') \xrightarrow{\cong} (A_{(L'',J)}, \lambda, j, \overline{\eta}).$$

This completes the verification that Φ is surjective.

Remark 8.2. The variety $Sh^0_{\mathbb{C}}$ has many equivalent descriptions which do not depend on a choice of a collection of representing lattices for the lattice classes in $\mathrm{Cl}_L(G)$:

$$\begin{aligned} Sh^0_{\mathbb{C}} &\cong G(\mathbb{Q})^+ \backslash (G(\mathbb{A}^\infty)/K^\infty \times \mathcal{H}) \\ &\cong G(\mathbb{Q})^+ \backslash (G(\mathbb{A}^\infty) \times G^+(\mathbb{R}))/K^\infty (K'_\infty)^+ \\ &\cong G(\mathbb{Q}) \backslash G(\mathbb{A})/K^\infty K'_\infty. \end{aligned}$$

Here, $K'_\infty \subset G(\mathbb{R})$ is the stabilizer of the complex structure I, and (K'_∞) is the intersection of K'_∞ with $G^+(\mathbb{R})$.

9. The irreducible representations of K^1_∞

The Hermitian symmetric domain $\mathcal{H}$ admits yet another description as the quotient

$$\mathcal{H} = SU(\mathbb{R})/K^1_\infty.$$

Here, SU is the norm 1 subgroup of U, whose R points are given by

$$SU(R) = \{g \in U(R) \,:\, \mathrm{Nr}(g) = 1\}$$

and K^1_∞ is the corresponding subgroup of K_∞. By Proposition 2.2, the group K^1_∞ is given by

$$(U((\epsilon V_\infty)^+_I) \times U((\epsilon V_\infty)^-_I))^{\det = 1}.$$

Therefore, the group K^1_∞ is abstractly isomorphic to $U(n-1)$.

The weights of the automorphic forms on $SU(\mathbb{R})$ are the dominant weights of irreducible representations of the maximal compact group K^1_∞ (all of the representations in this section are finite dimensional complex representations). We describe these irreducible representations in geometric terms.

Remark 9.1. Because I know no representation theory, some of the statements in this section might be flat out wrong. For this section and the next, should I be using irreducible reps of K_∞ instead of K^1_∞? I'm not sure that I care, because in the end, I am just trying to justify sections of a certain line bundle are instances of holomorphic automorphic forms...

To expedite notation, let $W = \epsilon V_\infty$, and let $W^\pm = W_I^\pm$. We shall regard each of these as complex vector spaces with complex structure given by I. We have the following isomorphism of compact Lie groups:

$$\begin{aligned} U(W^-) &\xrightarrow{\cong} (U(W^+) \times U(W^-))^{\det=1} = K_\infty^1 \\ g &\mapsto (\det(g)^{-1}, g). \end{aligned}$$

Using the "unitary trick", the irreducible representations of $U(W^-)$ are the same as the holomorphic irreducible representations of $GL = GL_{\mathbb{C}}(W^-)$. Fix a Borel subgroup B with maximal torus T and unipotent radical N. With respect to our preferred basis of W^-, we take the torus to be the group of diagonal matrices

$$z = (z_1, \ldots, z_{n-1}) = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_{n-1}. \end{bmatrix},$$

B the upper triangular matrices, and N the upper triangular matrices with 1's on the diagonal.

The Stiefel manifold. Define the Stiefel manifold $St = St(W^-)$ to be the the complex homogeneous space GL/N. It is the space of tuples $(F, (\vec{v}_i))$ where

$$F = (0 = W_0^- < W_1^- < \cdots W_{n-1}^- = W^-)$$

is the space of complete flags in W^-, and $\vec{v}_i \in W_i^-/W_{i-1}^-$ is a choice of non-zero vector for each i.

We may equally well regard a point in St as a sequence of equivalence classes of linearly independent vectors

$$([\vec{v}_1], \ldots, [\vec{v}_{n-1}])$$

where $\vec{v}_i$ lies in W^-, and where $\vec{v}_i$ is equivalent to $\vec{v}_i'$ if their difference may be expressed in terms of the (well defined) $i-1$ dimensional subspace W_{i-1}^- spanned by $\vec{v}_1, \ldots, \vec{v}_{i-1}$. The flag is then implicit in this data.

Let $\mathcal{V} = \mathcal{V}(W^-)$ be the space of holomorphic functions on St. Thus an element f of $\mathcal{V}$ may be regarded as function which associates to a collection of linearly independent vectors $(\vec{v}_i)$ in W^- a complex number

$$f(\vec{v}_1, \ldots, \vec{v}_{n-1})$$

which only depends on the equivalence classes $([\vec{v}_i])$. We shall refer to such a function as a *Stiefel function* on W^-.

The torus T acts on the right on St by

$$([\vec{v}_i]) \cdot z = ([z_i \vec{v}_i])$$

and the whole group GL acts on the left by

$$g \cdot ([\vec{v}_i]) = ([g\vec{v}_i]).$$

These two actions commute.

Borel-Weil theory. Let X be the lattice of characters of T. There is an associated cone of positive weights $X^+ \subset X$. We have $X^+ \cong \mathbb{N}^{n-2} \times \mathbb{Z}$ where the $\mathbb{Z}$ corresponds integer powers of the determinant character.

The quotient $\mathcal{F} = St/T$ is the flag manifold of complete flags in W^-. Define a line bundle $\mathcal{L}_\kappa$ over $\mathcal{F}$:

$$\begin{array}{c} \mathcal{L}_\kappa = St \times_T \mathbb{C}^\kappa \\ \downarrow \\ \mathcal{F} = St/T \end{array}$$

The group GL acts on $\mathcal{V}$ on the left by

$$(g \cdot f)(\vec{v}_1, \dots \vec{v}_{n-1}) = f(g^{-1}\vec{v}_1, \dots g^{-1}\vec{v}_{n-1}).$$

We can decompose the representation $\mathcal{V}$ by

$$\mathcal{V} = \bigoplus_{\kappa \in X} \mathcal{V}_\kappa$$

where we have

$$\begin{aligned} \mathcal{V}_\kappa &= H^0(\mathcal{F}, \mathcal{L}_{-\kappa}) \\ &= \{f \in \mathcal{V} \,:\, f(z_i \vec{v}_i) = \kappa(z)^{-1} f(\vec{v}_i)\}. \end{aligned}$$

In particular, since $\mathcal{F}$ is a projective variety, these representations are finite.

Theorem 9.2 (Borel-Weil)**.** If κ is positive, then $\mathcal{V}_\kappa$ is the irreducible representation of GL of dominant weight κ, otherwise $\mathcal{V}_\kappa = 0$.

10. Automorphic forms

Some vector bundles on $\mathcal{H}$. Given a positive weight $\kappa \in X^+$, we define a line bundle ω_κ over $\mathcal{H}$ by

$$\begin{array}{c} \omega_\kappa = SU(\mathbb{R}) \times_{K^1_\infty} \mathcal{V}_\kappa \\ \downarrow \\ \mathcal{H} = SU(\mathbb{R})/K^1_\infty. \end{array}$$

A section of ω_κ is a function f which associates to a compatible complex structure $J \in \mathcal{H}$ and a collection of $n-1$ linearly independent vectors $\vec{v}_i$ in W^-_J a complex number

$$f(J, (\vec{v}_i))$$

so that

1. f is a holomorphic Stiefel function in $\vec{v}_i$.
2. $f(J, (z_i\vec{v}_i)) = \kappa(z)^{-1} f(J, (\vec{v}_i))$ for all $z = (z_i) \in T$.

Automorphic forms. Let L' be a representative of a lattice class in $\mathrm{Cl}_L(G)$. The vector bundle ω_κ is naturally $\Gamma(L')$-equivariant.

Definition 10.1. A *weakly holomorphic automorphic form* for the congruence subgroup $\Gamma(L')$ of weight κ is a holomorphic section of ω_κ which is invariant under the $\Gamma(L')$ action. We shall denote the space of all such automorphic forms by $\mathcal{A}^{weak}_\kappa(L')$.

The weight κ weakly holomorphic automorphic forms may be regarded as the space of holomorphic sections

$$H^0(Sh^0_{\mathbb{C}}, \omega_\kappa) = \prod_{L' \in \mathrm{Cl}_L(G)} \mathcal{A}^{weak}_\kappa(L').$$

Some care must be taken though in taking this to be the definition, because the variety $Sh^0_{\mathbb{C}}$ typically has singularities.

A holomorphic automorphic form is a weakly holomorphic automorphic form that also satisfies some growth conditions. These conditions are equivalent to insisting that the form extend over a suitable compactification of $Sh^0_{\mathbb{C}}$. However, if B is a division algebra, the variety $Sh^0_{\mathbb{C}}$ is compact, and no growth conditions are necessary.

Remark 10.2. Classically, an automorphic form on a real group such as $SU(\mathbb{R})$ is (loosely speaking — there seem to be issues with the center because $\Gamma(L')$ is not necessarily contained in $SU(\mathbb{R})$) a K^1_∞-equivariant $\Gamma(L')$-invariant map

$$SU(\mathbb{R}) \to \mathcal{W}$$

for a K^1_∞-representation $\mathcal{W}$ satisfying two conditions:

(1) f satisfies a $Z(\mathfrak{sl})$ finiteness condition.
(2) f satisfies a growth condition.

It turns out that our holomorphy conditions constitute a particularly strong instance of $Z(\mathfrak{sl})$-finiteness. It is for this reason that we say that our automorphic forms are *holomorphic automorphic forms.* The growth conditions constitute a holomorphy condition at the cusps, but if $Sh^0_{\mathbb{C}}$ is compact, there are none.

Concretely, a holomorphic automorphic form f of weight κ may be regarded as a rule, which associates to a point

$$\underline{A} = (A, \lambda, j, \overline{\eta}) \in \mathcal{X}(\mathbb{C})$$

and a sequence of $n-1$ linearly independent vectors $(\vec{v}_i)$ in $(\epsilon t_A)^-_j$ (where t_A is the tangent space of A), a complex number

$$f(\underline{A}, (\vec{v}_i))$$

satisfying:

(1) f is holomorphic in $\underline{A} \in Sh^0_{\mathbb{C}}$ and the sequence $(\vec{v}_i)$.
(2) for fixed $\underline{A}$, f is a Stiefel function in $(\vec{v}_i)$.
(3) for $z = (z_i) \in T$, we have

$$f(\underline{A}, (z_i \vec{v}_i)) = \kappa(z)^{-1} f(\underline{A}, (\vec{v}_i)).$$

Weights in the determinant line. There is a natural line bundle ω over $Sh^0_{\mathbb{C}}$: its fiber over $\underline{A} = (A, \lambda, j, \overline{\eta})$ is given by

$$\omega_{\underline{A}} = [(\epsilon t_A)^+_j]^*,$$

the dual of the summand of the tangent space t_A of A at the identity.

We now demonstrate that if $\det \in X^+$ is the determinant character, then we have

$$\omega = \omega_{\det}.$$

Thus, for $k \in \mathbb{Z}$, holomorphic sections of $\omega^{\otimes k}$ give automorphic forms of weight $k \cdot \det$:

$$H^0(Sh^0_{\mathbb{C}}, \omega^{\otimes k}) = H^0(Sh^0_{\mathbb{C}}, \omega_{(k \cdot \det)}).$$

Indeed, consider the 1-dimensional representation χ of K^1_∞ given by

$$\begin{aligned} \chi : K^1_\infty = (U(W^+) \times U(W^-))^{\det = 1} &\to U(1) \\ (a, x) &\mapsto a. \end{aligned}$$

The representation χ has character $(-\det)$ when regarded as a representation of $U(W^-)$. It therefore coincides with the representation $\mathcal{V}_\kappa$, and the associated line bundle

$$\begin{array}{c} \tau = \coprod_{L'} \Gamma(L') \backslash (SU(\mathbb{R}) \times_{K^1_\infty} \mathbb{C}^\chi) \\ \downarrow \\ Sh^0_{\mathbb{C}} = \coprod_{L'} \Gamma(L') \backslash (SU(\mathbb{R}) / K^1_\infty). \end{array}$$

must coincide with the vector bundle $\omega_{-\det}$. On the other hand, the vector bundle τ is quite explicitly the vector bundle whose fiber over a point $\underline{A} = (A, \lambda, j, \overline{\eta})$ is the line $(\epsilon t_A)^+_j$. The result now follows by taking duals.

Therefore, a weakly holomorphic automorphic form f of weight $k \cdot \det$ may be regarded as a rule which associates to a point

$$\underline{A} = (A, \lambda, j, \overline{\eta}) \in Sh^0_{\mathbb{C}}$$

and a non-zero vector $\vec{v}$ in $(\epsilon t_A)^+_j$, a complex number

$$f(\underline{A}, \vec{v})$$

satisfying:

(1) f is holomorphic.
(2) for $z \in \mathbb{C}^\times$, we have

$$f(\underline{A}, z\vec{v}) = z^k f(\underline{A}, \vec{v}).$$