Barwick - K-theory + crystals

Note Title 4/16/2009

1968? Groth wrote letter to Tate

"Cohomology in char p"

Dix exposés; Inf cohomology

1974? Berthelot's Thesis

Berthelot - Ogus

HOW TO COMPUTE?

1981? Illusie (inspired by Deligne)

De Rham - Witt complex

1977: Spencer Bloch: use K-thy to construct a complex to compute crystalline coh.

[influenced by Mazur]

1996? Hesselholt: finished off Bloch's program.

<u>Fine</u> <u>point</u>: alg K-thy for rings

<u>Need</u>: (1) $K(-)$ to be <u>very</u> functorial

(2) $K(-)$ must have a universal property

(3) inputs need to be <u>more</u> <u>general</u> than abelian or exact cats.

<u>e.g.</u> X = scheme

$K(X)$ using Quillen's machine will NOT BE FUNCTORIAL ENOUGH

<u>Need</u>: Waldhausen K-thy

<u>Input</u>: sufficiently nice cats equipped w/ notion of w.e.

<u>Rule</u>: ALL SCHEMES ARE QUASI-COMPACT

e.g. X = scheme

$\mathrm{Perf}(\mathcal{O}_X)$: perfect cx's of $\mathcal{O}_X$-modules

w.e. = quasi-isos

$$K : \text{Wald Cats} \longrightarrow Sp_{\geq 0}$$

Connective Spectra
$\wr\wr$
∞-loop spaces

$$\pi_* K(\mathrm{Perf}(\mathcal{O}_{\mathrm{Spec}(R)})) = K_*(R)$$

Key point:

"Algebraic K-theory of schemes has transfers"

Structures on $K(X)$

(1) $\otimes^{\mathbb{L}}$: $\mathrm{Perf}(X) \times \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(X)$

$\Longrightarrow$ $$K(X) \wedge K(X) \longrightarrow K(X)$$

$K(X)$ is an E_∞-ring

$\Longrightarrow$ $K_*(X)$ is a ring.

(2) S = base scheme

X, Y S-schemes

$\boxtimes^{\mathbb{L}}$: $\mathrm{Perf}(X) \times \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(X \times_S Y)$

$$E\ ,\ F \longmapsto (\mathbb{L}pr_1^* E) \otimes^{\mathbb{L}} (\mathbb{L}pr_2^* F)$$

$\Rightarrow \quad \boxtimes : K(X) \wedge K(Y) \longrightarrow K(X \times_S Y)$

(3) $f : Y \to X$

$$\mathbb{L}f^* : \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(Y)$$

This defines: functor

$$Sh^{op} \longrightarrow Sp_{\geq 0}$$
$$X \longmapsto K(X)$$

(4) $f : Y \to X$

$$\mathbb{R}f_* : \mathrm{QCoh}(\mathcal{O}_Y) \longrightarrow \mathrm{QCoh}(\mathcal{O}_X)$$

complexes of $\mathcal{O}_X$-mods w/ quasi-coherent coh

If f is flat, proper

$$\Rightarrow \mathbb{R}f_* : \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(X)$$

$$\mathrm{Sch}_{\mathrm{fl,pr}} \xrightarrow{K} \mathrm{Sp}_{\geq 0}$$

co-variant!

subcat of flat and proper maps

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⑤ Base-Change thm:

$$\begin{tikzcd}
Y' \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "f"] \\
X' \arrow[r, "g"'] & X
\end{tikzcd}
\qquad f, g \in \mathrm{Sch}_{\mathrm{fl,pr}}$$

$$\begin{tikzcd}
\mathrm{Perf}(X') \arrow[r, shift left, "\mathbb{L}f^{*}\mathbb{R}g_{*}"] \arrow[r, shift right, "\mathbb{R}g_{*}\mathbb{L}f^{*}"'] & \mathrm{Perf}(Y)
\end{tikzcd}$$

Thm (SGA6)

$$\mathbb{L}f^{*}\mathbb{R}g_{*}E \simeq \mathbb{R}g_{*}\mathbb{L}f^{*}E$$

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⑥ $\mathbb{L}f^*(E \otimes^{\mathbb{L}} F) \simeq \mathbb{L}f^*E \otimes \mathbb{L}f^*F$

<u>But:</u>

⑦ f flat, proper

$$(\mathbb{R}f_* E) \otimes^{\mathbb{L}} F \xrightarrow[\sim]{} \mathbb{R}f_*(E \otimes \mathbb{L}f^*F)$$

$\Longrightarrow$ $f^* : K(X) \longrightarrow K(Y)$

is a map of E_∞-alg's

$f_* : K(Y) \longrightarrow K(X)$

is a morphism of $K(X)$-mods

$\Longrightarrow$ K-thy is a a spectrally-valued Green Functor.

1. Define $\mathcal{B}_{/S}$

Burnside cat (enriched in spectra)

Objects: S-schemes (quasi-compact)

Maps: $\mathcal{B}_{/S}(X, Y)$ spectrum.

Construction of spectrum

$\underline{\mathcal{B}}_{/S}(X, Y)$ = category

Objects: $X \leftarrow U \xrightarrow{\text{flat, proper}} Y$ of S-schs

symmetric monoidal w/ $\amalg$

$$\underline{\underline{B}}_{/S}(X,Y) \times \underline{\underline{B}}_{/S}(Y,Z) \to \underline{\underline{B}}_{/S}(X,Z)$$

$$\begin{tikzcd}[column sep=small]
 & & U \times_Y V \arrow[dl] \arrow[dr] & & \\
 & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

$$\mathcal{B}_{/S}(X,Y) = \Omega B N^{iso} \underline{\underline{B}}_{/S}(X,Y)$$

connective spectrum

$\mathcal{B}_{/S}$ is a $Sp_{\geq 0}$-enriched cat.

<u>Rmk:</u> If you pass to π_0 get Q-const

Hopes: Replace schemes w/ gps $\Longrightarrow$ Segal Conj

Def: A (spectrally-valued) Mackey functor is a funct of spectral cat's

$$M : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$

What does this amount to?

$$Sch^{op}_{/S} \longrightarrow \mathcal{B}_{/S}$$

$$(Y \to X) \longmapsto \left(\begin{tikzcd} & Y \arrow[dl] \arrow[dr, equal] & \\ X & & Y \end{tikzcd} \right)$$

$$Sch_{fl,pr/S} \longrightarrow \mathcal{B}_{/S}$$

$$(Y \to X) \longmapsto \left(\begin{tikzcd} & Y \arrow[dl, equal] \arrow[dr] & \\ Y & & X \end{tikzcd} \right)$$

$\underline{M}$: Mackey functor

$$M^* : Sch^{op}_{/S} \longrightarrow Sp_{\geq 0}$$

$$M_* : Sch_{fl,pr/S} \longrightarrow Sp_{\geq 0}$$

$$f : Y \longrightarrow X$$

$$M(f) = f^*$$

$$M(f) = f_*$$

$\underline{\underline{\text{Functoriality}}} \iff$ base change

$$f^* g_* \simeq g_* f^*$$

Lemma: $K(-)$ is a Mackey functor.

Fact: $\{\text{Mackey functors}\} = \text{Sym}_{\text{Cat}}$ mon

Comm Monoids = "green functors"

Thm K-theory will turn out to be a Green functor

$$M, N : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$

get

$$\begin{tikzcd}
(X,Y) \arrow[d, mapsto] & \mathcal{B}_{/S} \times \mathcal{B}_{/S} \arrow[r, "M \times N"'] \arrow[d, "\text{sym mon structure}"'] \arrow[drr, "M \boxtimes N"] & & Sp_{\geq 0} \times Sp_{\geq 0} \arrow[d, "1"] \\
X \times_S Y & \mathcal{B}_{/S} \arrow[rr, dashed, "\text{Left Kan ext}", "M \otimes N"'] & & Sp_{\geq 0}
\end{tikzcd}$$

Day convolution

$$(M \otimes N)(Z) = \text{"colim"}_{X \times_S Y \to Z} \; M(X) \wedge N(Y)$$

Green Functor$/S$: comm monoid in Mack$/S$

Rmk: Could consider algebras over arbitrary operads

UNPACK THE STRUCTURE:

$A : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$ green functor

bunch of maps

"$A(X) \wedge A(Y) \longrightarrow A(X \times_S Y)$"

Lem: K-thy is a green funct.

Aside

"One should do alg geometry of GREEN FUNCTORS"

Witt complexes redux

$$\mathrm{Set}^f_{\mathbb{Z}} = \text{finite } \mathbb{Z}\text{-sets}$$

Can form a "Burnside cat"

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ S & & T \end{tikzcd} \qquad \mathcal{B}_{\mathbb{Z}}^{op}$$

Φ = poset of positive integers ordered by division

$$m \to n \iff m \mid n$$

Main point: $\Phi \simeq$ transitive finite $\mathbb{Z}$-sets

$\mathcal{B}_\Phi$ = Burnside cat of these

Fact:

$$\pi_0 \mathcal{B}_\Phi \longrightarrow Ab$$

① $\forall\ n \geq 1 \rightsquigarrow W_n$

② $m|n$

$$F_{m|n} : W_n \longrightarrow W_m$$

③ $m|n$

$$V_{m|n} : W_m \longrightarrow W_n$$

Subject to:

① $\forall \quad \ell | m | n$

$$F_{\ell|n} = F_{\ell|m} \circ F_{m|n}$$

$$V_{\ell|n} = V_{m|n} \circ V_{\ell|m}$$

② $\ell | n \;, \quad m | n$

$$F_{\ell|n} \circ V_{m|n} = \sum_{\epsilon \in (\mathrm{lcm}(\ell,m)\mathbb{Z}/m\mathbb{Z})} V_{\gcd(\ell,m)|\ell} \circ F_{\gcd(\ell,m)|m}$$

$$F_{n/\ell} \circ V_{n/m} = V_{n/m} \circ F_{n/\ell}$$

$$\gcd(n/m, n/\ell) = 1$$

Compare w/ Witt vectors

Can characterize:

$\mathbb{W}(R) =: W_n^{Big}(R)$ as connected Green functor

w/ teichmüller lifts associated to R

Fix $R = $ ring

Def: A Comm Green functor

$$M : \mathcal{B}_{\mathbb{Z}} \longrightarrow Sp_{\geq 0}$$

(with connective) and

$$\pi_0 M : \pi_0 \mathcal{B}_{\mathbb{Z}} \longrightarrow Ab$$

$$\pi_0 M = \mathbb{W}(R)$$

is called "Grand Witt Complex"

Initial Object in this ∞-cat of Grand Witt Complexes

is the Grand de Rham-Witt-cx.

Bloch:

Green functors $\xrightarrow[\text{curves}]{}$ Grand Witt Cx

A = Green functor $/ S$

$$TA : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$ Green functor

$$X \longmapsto A(X[\varepsilon])$$

$$T_0 A \longrightarrow TA \longrightarrow A$$

"tangent space of identity"

in char p, $\mathbb{G}_a$ = alg gp in char p

Need to consider entire inf. nbhd of id.

Main thm:

$$\Theta_m A : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$

$$X \longmapsto A\left(X[t]/t^m\right)$$

Mackey functor

$$C_m A \longrightarrow \Theta_m A \longrightarrow A$$

$$\hat{C}A = \operatorname{holim}_{m \to \infty} \Omega C_m A$$

"curves of A"

jets of curves through origin

$$\varphi_{m/n} : X[u]/u^n \longrightarrow X[t]/t^m$$

$$u^{n/m} \longleftarrow t$$

comp w/ identity on A

$$\phi^*_{m/n} : \Theta_m A \longrightarrow \Theta_n A$$

$$(\phi_{m/n})_* : \Theta_n A \longrightarrow \Theta_m A$$

comp w/ $\cdot m$ on A

On fibers, roles get reversed

$\phi^*\big|_{\text{fiber}}$ = map of modules

$\phi_*\big|_{\text{fiber}}$ = map of rings

$$V_{m/n} : C_m A \longrightarrow C_n A$$

$$F_{m/n} : C_n A \longrightarrow C_m A$$

<u>Thm</u>: (Bloch–Hesselholt) R = smooth ring

$A = K$

$\hat{C}K(R)$ is a green funct on $\mathbb{Z}$

$\hat{C}K(R)$ is the Grand de-Rham wrt complex.

Note f gets a diff'l

$\pi_* \hat{C}K(R)$ = classical big de Rham wrt cx.
