Isaacson:

Note Title 3/19/2009

<u>Top</u> (compactly generated top'l spaces)

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad \underline{\text{homotopic}}$$

if

$$\begin{tikzcd} X \arrow[d, "h_0"] \arrow[dr, "f"] & \\ X \times I \arrow[r, "\exists h"] & Y \\ X \arrow[u, "h_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

<u>Recall:</u> $\pi_n(X, x_0) = [(D^n, S^{n-1}), (X, x_0)]$

$f: X \to Y$ is a w.e.

if

$$\pi_n(X, x_0) \xrightarrow[\cong]{} \pi_n(Y, f(x_0))$$

$\forall$ n, all x_0

$$Ho(Top) = Top[(w.e.)^{-1}]$$

$X =$ space

$\exists$ CW cx γX

and a w.e.

$$\gamma X \xrightarrow[\simeq]{} X$$

$$Ho(Top)(X, Y) = [\gamma X, \gamma Y]$$

Analogy: Chain cx's /R

Top		Ch(R)	
Ho(Top)	$\longleftrightarrow$	$\mathcal{D}(R)$	derived cat
homotopy	$\longleftrightarrow$	chain htpy	
w.e.	$\longleftrightarrow$	homology iso	
CW-approx	$\longleftrightarrow$	projective resolution	
"built out of (D^n, S^{n-1})"	$\longleftrightarrow$	"built out of R"	

Δ = cat of finite non-empty ordered sets, order preserving maps

Iso classes:

$$[n] = \{0 < 1 < \cdots < n\}$$

face map $\quad \delta^i : [n-1] \longrightarrow [n] \quad$ omit i

deg maps $\quad s^i : [n] \longrightarrow [n-1]$

$\Delta^n \in \underline{Top} \qquad \Delta^n \subseteq \mathbb{R}^{n+1}$

$$\left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_0 + \cdots + x_n = 1 \\ x_i \geq 0 \end{array} \right\}$$

$[n] \longmapsto \Delta^n$

$\Delta \longrightarrow \underline{Top}$

Define $\underline{S}$ = cat of simp sets

$= \mathrm{Fun}(\Delta^{op}, \underline{Set})$

$$|-| : \underline{S} \rightleftarrows \underline{Top} : \mathrm{Sing}$$

geometric realization

singular complex

$Z \in \mathrm{Top}$

$$(\mathrm{Sing}\, Z)_n = \mathrm{Hom}(\Delta^n, Z)$$

$$\shortparallel$$

$$(\mathrm{Sing}\, Z)([n])$$

$$|X| = \coprod_n X_n \times \Delta^n \Big/ \begin{array}{c} (f^* x_n, y) \\ \wr \\ (x_n, f_* y) \end{array}$$

$X \in \underline{S}$

$\Delta[n] :=$ simplicial set represented by $[n]$

$$[m] \longmapsto \mathrm{Hom}([m],[n])$$

Check:

$$|\Delta[n]| = \Delta^n \quad \text{(abstract nonsense)}$$

$$|X \times Y| \cong |X| \times |Y| \quad \text{(tricky)}$$

Two maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are homotopic

if:

$$\begin{tikzcd}
X \arrow[d, "\mathrm{Id}\times d^1"] \arrow[drr, bend left] & & \\
X \times \Delta[1] \arrow[rr, dashed] & & Y \\
X \arrow[u, "\mathrm{Id}\times d^0"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

Catch!

homotopy is **Not** an equiv. relation.

Defi:

$Y \in \underline{S}$ has the <u>homotopy extension property</u> if:

$\forall$ monos:

$$A \hookrightarrow B$$

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A \times \Delta[1] \arrow[r] \arrow[drr, bend right] & B \times \Delta[1] \arrow[dr, dashed, "\exists"'] & \\
 & & Y
\end{tikzcd}$$

Check: If Y has the HEP, then htpy is an equivalence relation on $\underline{S}(X, Y)$

Thm (Kan)

Y has HEP iff

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & Y \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Here: $\Lambda^i[n]$ is the boundary of $\Delta[n]$ w/ $d^i \Delta[n-1]$ removed

$$\Lambda^i[n]([m]) = \left\{ [m] \xrightarrow[f]{} [n] \;\middle|\; \begin{array}{l} \text{s.t. } f \text{ factors} \\ \text{thru } d^j \\ \text{for } j \neq i \end{array} \right\}$$

Such Y are called <u>Kan</u> <u>cx's</u>

<u>Thm</u> (Kan)

$X \in \underline{S}$

$\exists$ Kan complex $\tilde{X}$

and a map $X \to \tilde{X}$

s.t. $|X| \xrightarrow[\simeq]{} |\tilde{X}|$

<u>e.g.</u> $X \to \operatorname{Sing}|X|$

<u>Kan complexes</u>

Define:

$Ho(\underline{S})$: Objects = Ob $\underline{S}$

$$Ho(\underline{S})(X,Y) = [\tilde{X}, \tilde{Y}]$$

maps between
Kan cx's
mod homotopy

Thm:

$$|-| : \underline{S} \rightleftarrows \underline{Top} : Sing$$

descends to an equivalence

$$|-| : Ho(\underline{S}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} Ho(\underline{Top}) : Sing$$

Simplicial sets give a way to give rigid constructions of spaces

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II: <u>examples</u>

$$Ab(\underline{S}) \longrightarrow \underline{S}$$

simplicial ab gp

$$M_\bullet \in Ab(\underline{S})$$

$$\downarrow$$

$$C_*(M_\bullet) \in \text{Chain cx's}$$

$$\cdots \longrightarrow M_n \xrightarrow{\partial} M_{n-1} \longrightarrow \cdots \longrightarrow M_0$$

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

$\exists$ subcomplex

$$NC_*(M) \subset C_*(M_\bullet)$$

normalized chains

$$\begin{tikzcd}
\cdots \arrow[r] & \bigcap_{i=0}^{n-1} \ker d_i \arrow[r, "(-1)^n d_n"] \arrow[d, hook] & \bigcap_{i=0}^{n-1} \ker d_i \arrow[r] \arrow[d, hook] & \cdots & NC_*(M) \\
\cdots \arrow[r] & M_n \arrow[r] & M_{n-1} \arrow[r] & \cdots & C_*(M)
\end{tikzcd}$$

$$NC_* : \mathrm{Ab}(\underline{S}) \longrightarrow \mathrm{Ch}(\mathbb{Z})_{\geq 0}$$

equivalence of cats

(Dold – Kan)

Furthermore:

(1) $M \in \mathrm{Ab}(\underline{S}) \Longrightarrow M$ is a Kan cx

(2) $\pi_n(M) \cong H_n(NC_*(M))$

Applications:

$$S^n := \Delta[n]/\partial\Delta[n] \in \underline{S}$$

$$\downarrow$$

$$K(\mathbb{Z},n) = \mathbb{Z}\{S^n\}/\mathbb{Z}\{pt\} \in Ab(\underline{S})$$

$$\pi_i(K(\mathbb{Z},n)) = \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$$

Furthermore

$$|K(\mathbb{Z},n)| = \text{top'l ab sp.}$$

G = gp $\rightsquigarrow$ $K(G,1)$

G = cat w/ one object.

Suppose $\mathcal{C}$ = ctgy

The nerve of $\mathcal{C}$

$N\mathcal{C} \in \underline{S}$

$$(N\mathcal{C})_n = \mathrm{Func}([n], \mathcal{C})$$

here: $[n]$ is the cat

$$0 \to 1 \to \cdots \to n$$

$B\mathcal{C} = |N\mathcal{C}|$ "classifying space"

NG :

(1) $\pi_i NG = \begin{cases} 0, & i \neq 1 \\ G, & i = 1 \end{cases}$

(2) NG is a Kan complex

BG = classifying space of G

$$H_*(NG) = H_*(G;\mathbb{Z})$$

In fact $C_*(\mathbb{Z}[NG])$

$\parallel$

Bar resolution of G

$$BG = NG$$

$$EG = N\left(\begin{array}{l}\text{nerve of gpoid} \\ \text{objects} = G \\ \text{morphisms}: \quad g \xrightarrow[hg^{-1}]{} h\end{array}\right)$$

$\circlearrowleft g$

with A

G-action

$$EG \longrightarrow BG = EG/G$$

universal principle G-bundle.

$$X \in S$$

$G \circlearrowright X$

left action

$$X^G = \mathrm{Hom}_G(*, X)$$

$$X_G = * \times_G X$$

<u>Compare</u>

M = G-mod

$$M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$$

$$M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$$

Can compute gp homology/coh

by taking $\mathbb{Z}[G]$-free resolutions

of $\mathbb{Z}$

EG

$$X^{hG} = \mathrm{Hom}_G(EG, X)$$

"htpy fixed points"

$$X_{hG} = EG \times_G X$$

"htpy orbits"

<u>Note:</u> $(*)_{hG} = BG$

Htpy thy version of gp homology, gp cohomology

$$H_p(G, H_q(X)) \Rightarrow H_{p+q}(X_{hG})$$

Serre spectral sequence
associated to:

$$\begin{tikzcd} X \arrow[r] & X \times_G EG \arrow[d] \\ & BG \end{tikzcd}$$

$$\underline{Top}^{\Delta^{op}} = \text{simplicial spaces}$$

$$\uparrow \ |\text{-}| \quad \text{levelwise realization}$$

$$\underline{S}^{\Delta^{op}} = \mathrm{Func}(\Delta^{op} \times \Delta^{op}, \underline{Set})$$

bisimplicial set

$$\begin{tikzcd} \underline{S}^{\Delta^{op}} \arrow[r, "|-|"] \arrow[d, "\mathrm{diag}"] & \underline{Top}^{\Delta^{op}} \arrow[d, "|-|"] \\ \underline{S} \arrow[r, "|-|"'] & \underline{Top} \end{tikzcd}$$

w.e. = levelwise w.e.

$X \in \underline{S}^{\Delta^{op}}$

$$\coprod X_{\bullet n} \times \Delta[n] /\sim$$

this is a simplicial set

$$\Big\| $$

$$\mathrm{diag}(X_{\bullet\bullet})$$

$\underline{S}^{\Delta^{op}}$ can do "htpy thy"

$X_{\bullet\bullet} \longrightarrow Y_{\bullet\bullet}$ is a w.e.

iff

$X_{\bullet,n} \longrightarrow Y_{\bullet,n}$ is a w.e. $\forall n$

Miracle: diag is htpy invariant.

$X_\bullet \in \underline{\mathrm{Top}}^{\Delta^{op}}$

$\rightsquigarrow \quad |X_\bullet| = \coprod X_n \times \Delta^n \Big/ \sim$

Not htpy invariant

Segal Cat's and coh thy;

Appendix: explains how to make 1-1 htpy invariant.
