K-thy of fields

Note Title 4/28/2009

Back to K-thy

Summary of some K-thy calculations

$(\mathcal{C}, \oplus)$ symm monoidal cat

$$K(\mathcal{C}) = Gp(B\mathcal{C}, \oplus)$$

[$\mathcal{C}$ = top'l cat. can still form $B\mathcal{C}$]

Moreover! $K(\mathcal{C})$ is an infinite loop space

get a spectrum $\underline{K}(\mathcal{C})$, $\Omega^\infty \underline{K}(\mathcal{C}) = K(\mathcal{C})$

From now on! I will write $K(\mathcal{C})$ for $\underline{K}(\mathcal{C})$

$$\mathcal{C} = (Proj(R)^{f.g}, \oplus)$$

$$\Rightarrow K(\mathcal{C}) = K(R)$$

other examples

$$K(\text{finite sets}, \amalg) = \mathbb{S} = \Sigma^{\infty} S^0$$ sphere spectrum

$$K(\mathrm{Vect}_{\mathbb{C}}, \oplus) = ku$$ connective complex K-thy spectrum

$$K(\mathrm{Vect}_{\mathbb{R}}, \oplus) = ko$$

[associated coh thy: $ku^0(X) = \dfrac{\mathbb{Z}\{\text{vector bundles}/X\}}{[E_1] + [E_2] = [E_1 \oplus E_2]}$]

X: finite CW cx

Angelica:

$$\Omega^{\infty} K(R) = BGL_{\infty}(R)^{+} \times K_0(R)$$

Analog:

$$BGL_n(\mathbb{C}) \simeq BU(n)$$

$$\Omega^{\infty} ku = BU \times \mathbb{Z}$$

$$\pi_* ku = \pi_* BU \times \mathbb{Z}$$

computed by Bott!

$\mathbb{Z}$ 0 $\mathbb{Z}$ 0 $\mathbb{Z}$ 0 - - - - -

$\pi_* \, ko$: $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ ----

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$\mathbb{C}$
$\Big| C_2$
$\mathbb{R}$

Action of C_2 on $\pi_* ku$

$\mathbb{Z}$ 0 $\mathbb{Z}$ 0 $\mathbb{Z}$ 0 ----
$+1$ $\quad -1$ $\quad +1$

get a map

$$\pi_* ko \longrightarrow (\pi_* ku)^{C_2} = \begin{cases} \mathbb{Z}, & * \equiv 0 \ (4) \\ 0, & o/w \end{cases}$$

other $\mathbb{Z}/2$'s come from "spectral sequence"

$$H^s(C_2, \pi_t ku) \Rightarrow \pi_{t-s} ko$$

concentrated for $t-s \geq 0$

Alg K-thy of $\mathbb{R}, \mathbb{C}$??

$$K_1(\mathbb{C}) \cong \mathbb{C}^\times$$

$$\cup \qquad \uparrow \text{ roots of unity}$$

$$K_1(\mathbb{C})_{tor} \cong \mu \cong \mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/n$$

$$\circlearrowleft \; C_2$$

Thm:

$$\left(K_{2i-1}(\mathbb{C})\right)_{tor} = \mu(\mathbb{C})(i)$$

$$\circlearrowleft \; C_2 \qquad (g, z) \mapsto (g^i z)$$

$$K_{2i}(\mathbb{C})_{tor} = 0$$

E is a spectrum

$$E \xrightarrow[n]{} E \longrightarrow E/n$$

LES $\pi_i E \xrightarrow[\cdot n]{} \pi_i E \longrightarrow \pi_i E/n \to \pi_{i-1} E$

$$\pi_*(K(R)/n) =: K_*(R, \mathbb{Z}/n)$$

$$K_{2i}(\bar{F}; \mathbb{Z}/n) \xrightarrow[\cong]{} K_{2i-1}(\bar{F})_{n\text{-tor}}$$

$$\left[\ \hat{E} = \varprojlim_{n} E/n \qquad \underline{\text{profinite}}\ \underline{\text{comp}}\ \right]$$

$$K_*(R; \hat{\mathbb{Z}}) := \pi_*\left(\widehat{K(R)}\right)$$

<u>We</u> <u>see</u>:

$$\begin{array}{ccc} K_{2i}(\mathbb{C}) & \cong & \pi_{2i}\,\widehat{ku} \\ \cong & & \\ \hat{\mathbb{Z}}(i) & & \end{array} \qquad \text{as } \mathrm{Gal}(\mathbb{C}/\mathbb{R})\text{-mods.}$$

$$\begin{array}{c} \bar{F} \\ \mid\ G \\ F \end{array} \qquad \text{separably closed}$$

$$\begin{tikzcd} K_1(F) \arrow[r, phantom, "="] & F^\times \\ K_1(F)_{tor} \arrow[u, hook] \arrow[r, phantom, "="] & \mu(F) \arrow[u, hook] \end{tikzcd} \quad \circlearrowleft\ G$$

<u>Thm</u> (Suslin) [Quillen did $\mathbb{F}_p$]

$$K_{2i-1}(\bar{F})_{tor} = \mu(\bar{F})(i)$$

$$K_{2i}(\bar{F})_{tor} = 0$$

$G \circlearrowright \mu(\bar{F})(i)$ $\quad (g, z) \longmapsto g^{\vartheta}(z)$

<u>Naive Guess</u>:

$$K_*(F) = K_*(\bar{F})^G$$

<u>Less Naive Guess</u>

$$H^s(G; \pi_t K(\bar{F})) \Rightarrow \pi_{t-s} K(F)$$

"Quillen-Lichtenbaum Conj" — this converges for $t-s \geq d$

$$e_i: K_{2i-1}(F) \longrightarrow \mu(\bar{F})(i)^G \cong \mathbb{Z}/w_i(F)$$

$F = \mathbb{F}_q$ Finite fld $q = p^n$

$$K_{2i-1}(\overline{\mathbb{F}}_q) = \mu(\overline{\mathbb{F}}_q)_{(i)} = \varinjlim_k \mathbb{Z}/(q^k-1)$$

$$= \mathbb{Q}/\mathbb{Z}[1/p]\ (i)$$

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} = \mathrm{Frob}_q$$

$$\zeta_{q-1} \in \mathbb{F}_q^\times$$

$$\zeta_{q^k-1} \in \mathbb{F}_{q^k}^\times$$

$$H^i(\hat{\mathbb{Z}}\ ;\ \mu(\overline{\mathbb{F}}_q)(i))$$

$$\|$$

$$H^*\Big(\ \underset{0}{\mu(\overline{\mathbb{F}}_q)} \xrightarrow[\mathrm{Frob}_q^i - 1]{} \underset{1}{\mu(\overline{\mathbb{F}}_q)}\ \Big)$$

$$\|$$

$$\begin{cases} \mathbb{Z}/q^i-1, & * = 0 \\ 0, & \text{o/w} \end{cases}$$

Rmk:

$$\left(\mathbb{Z}/q^i-1\right)_{(\ell)}$$

may be computed:

$$K_{2i-1}(\mathbb{F}_q;\mathbb{Z}_\ell) \cong \left(\mathbb{Z}/q^i-1\right)_{(\ell)} = \mathbb{Z}_\ell/q^i-1$$

If $\langle q \rangle = \mathbb{Z}_\ell^\times$

$$\cong \begin{cases} \mathbb{Z}/\ell^j & \text{if } i = (\ell-1)\ell^{j-1}s \\ & \quad \ell \nmid s \\ 0 & \text{o/w} \end{cases}$$

aside

$$\begin{tikzcd} \pi^s_{2i-1} S \arrow[r] \arrow[rrd, bend right, "e\text{-invariant}"'] & K_{2i-1}(\mathbb{F}_q) \arrow[r] \arrow[d, phantom, "\cong"] & K_{2i-1}(\overline{\mathbb{F}}_q) \arrow[d, phantom, "\cong"] \\ & (\operatorname{Im} J)_{2i-1} \arrow[r, hook] & \mathbb{Q}/\mathbb{Z}_{(p)} \end{tikzcd}$$

Rmk:

$$\frac{B_i}{i} = \frac{c_i}{d_i} \qquad j = \nu_\ell(d_i)$$

Related to $\qquad \zeta(1-n) = -\frac{B_n}{n}$

Note:

$$K_{2i-1}(\mathbb{F}_q; \mathbb{Z}_p) = 0$$

$$K_0(\mathbb{F}_q; \mathbb{Z}_p) = \mathbb{Z}_p$$

Localization sequence (Inna's talk)

$\mathcal{O}$ = dedekind domain

F = fld of frac

$$\cdots \to \bigoplus_{\mathfrak{p}} K_i(\mathcal{O}/\mathfrak{p}) \longrightarrow K_i(\mathcal{O}) \longrightarrow K_i(F) \longrightarrow \bigoplus_{\mathfrak{p}} K_{i-1}(\mathcal{O}/\mathfrak{p}) \to \cdots$$

low degrees

$$\begin{array}{c} F \\ | \\ \mathbb{Q} \end{array} \quad \text{finite}$$

$$\begin{tikzcd}
\bigoplus_{v \text{ finite}} K_1(k_v) \arrow[r, "0"] \arrow[d, equal] & K_1(\mathcal{O}_F) \arrow[r, hook] \arrow[d, equal] & K_1(F) \arrow[d, equal] \arrow[ddd, bend left, to path={-- ([xshift=-40ex]\tikztotarget.north)}] \\
\bigoplus k_v^\times & \mathcal{O}_F^\times & F^\times \\
\bigoplus_{v \text{ finite}} K_0(k_v) \arrow[r] & K_0(\mathcal{O}_F) \arrow[r, two heads] \arrow[d, "\cong"] & K_0(F) \arrow[d, equal] \\
\bigoplus_{v \text{ finite}} \mathbb{Z} \cong \text{Frac Ideals} \arrow[u, equal] & Cl(F) \oplus \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

$$0 \to \underbrace{K_2(\mathcal{O}_F)}_{\text{finite}} \to K_2(F) \longrightarrow \bigoplus K_1(k_v) \longrightarrow 0$$

$$\bigoplus K_1(k_v) = \bigoplus k_v^\times$$

Thm! (Tate) If F contains m^{th} roots of unity

$$K_2(F)/m \cong Br(F)_{m\text{-tor}}$$

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e.g.

$$K_2(\mathbb{Q})/2 \xrightarrow{\text{“Hilbert symbol”}} \bigoplus_p \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2 \longrightarrow 0$$

$$Br(\mathbb{Q})_{2\text{-tor}}$$

Thm (Soulé) $K_n(\mathcal{O}_F) \hookrightarrow K_n(F)$

localization sequence splits

$$K_{2i-1}(\mathcal{O}_F) \cong K_{2i-1}(F)$$

$$0 \to K_{2i}(\mathcal{O}_F) \to K_{2i}(F) \to \bigoplus_v \mathbb{Z}/q_v^i - 1 \to 0$$

Thm: (Mitchell) (Weibel-Rognes) $w_{2k}(\mathbb{Q}) = d_k$ $\left(\frac{B_{2k}}{4k}\right) = \frac{c_k}{d_k}$

n	$8i$	$8i+1$	$8i+2$	$8i+3$
$K_n(\mathbb{Z})$	0 ?	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_{2i+1}$	$\mathbb{Z}/d_{2i+1}$

$8i+4$	$8i+5$	$8i+6$	$8i+7$
0 ?	$\mathbb{Z}$	$\mathbb{Z}/c_{2i+2}$	$\mathbb{Z}/d_{2i+2}$

(Vandiver conj)

$$\zeta(1-2k) = -\frac{B_{2k}}{2k} = (-1)^k \frac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|}$$ up to factors of 2

Thm: F totally real

$$\zeta_F(1-2k) = (-1)^{kr_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|}$$ up to factors of 2

Local calculations

"étale K-thy"

$\ell \neq p$

$$K_i(\mathbb{Z}_p ; \mathbb{Z}_\ell) \cong K_i(\mathbb{F}_p ; \mathbb{Z}_\ell)$$

$$W_i(\mathbb{Q}_p) = \begin{cases} p^i - 1, & (p-1) \nmid i \\ (p^i - 1)p^j, & i = (p-1)p^{j-1}s \end{cases}$$

$$w_i^{(\ell)}(\mathbb{Q}_p) = w_i^{(\ell)}(\mathbb{F}_p)$$

$$w_i^{(p)}(\mathbb{Q}_p) = \text{p-fact of denom} \left(\frac{B_i}{2i}\right)$$

Thm (Hesselholt-Madsen) (Bökstedt-Madsen)

$$\begin{array}{c} K \\ \mid d \\ \mathbb{Q}_p \end{array}$$

(Predicted by Quillen-Lichtenbaum)

$$K_n(\mathcal{O}_K ; \mathbb{Z}_p) \cong K_n(K ; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}/w_i^{(p)}(K) & n = 2i \\ (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_i^{(p)}(K) & n = 2i-1 \end{cases}$$

$n \geq 2$

e.g.

$$K(W(\mathbb{F}_q))^\wedge_p \simeq \Sigma\, ku^{d-1} \vee \Sigma^3 ku \vee j \vee \Sigma j$$

$q = p^d$

Hesselholt - Madsen method of computation

Bloch: compute crystalline coh via K-thy

F, V, d $\qquad W\Omega_R = \Omega C_* K(R)$

Idea: Compute K by trying to understand $C_* K(R)$

Slogan LES!

$$\begin{tikzcd} W\Omega_R \arrow[rr, "d"] & & W\Omega_R \arrow[dl] \\ & H_{cris} \arrow[ul] & \end{tikzcd}$$

$$\begin{tikzcd} W\Omega_R \arrow[rr, "F-1"] & & W\Omega_R \arrow[dl] \\ & TC \arrow[ul] & \end{tikzcd}$$

"TC is K"

$$T K(R) \longrightarrow K(R[x]/x^2) \longrightarrow K(R) \longrightarrow \Sigma^{-1} T K(R)$$

"Shrodon"

$R[x]/x^2$ = square zero extension

$\cong$

$R \ltimes R$

R = ring
M = R-module

$R \ltimes M$ $\quad (r_1, m_1)(r_2, m_2) = (r_1 r_2, \ r_1 m_2 + r_2 m_1)$

Ring spectra

$$R[x]/x^n \longrightarrow R$$

infinitesimal thickening

E = ring spectrum (connective)

$$E \longrightarrow H\pi_0 E$$

infinitesimal thickening

idea: the more highly connected this is the more infinitesimal the thickening

$X \in \mathrm{Top}_*$, M a bimodule

$$HR \ltimes HM \wedge X_+$$

Can take K(ring spectra)

$$K(E) = K(E\text{-Mod}_{\text{finite, free}}, \vee)$$

$\nwarrow$ topological symmetric monoidal cat

$$K(HR) = K(R)$$

$$T_0 K(HR)_X \longrightarrow K(HR \ltimes HR \wedge X) \longrightarrow K(HR)$$

$$F : X \longmapsto K(HR \ltimes HR \wedge X)$$

$$\mathrm{Top}_* \longrightarrow \mathrm{Spectra}$$

(1) $F(*) = K(R)$

(2) F takes w.e's to w.e's

(3) colimit axiom

$$f : \mathbb{R} \to \mathbb{R}$$

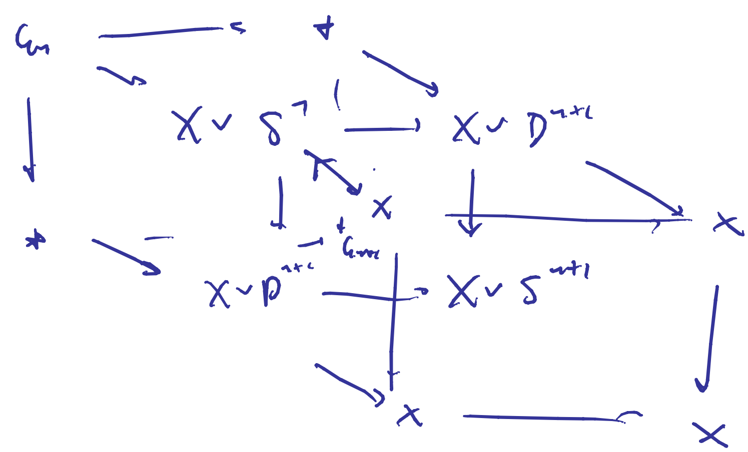
$$f' = \lim_{n\to\infty} \left(\tfrac{1}{n}\right)^{-1}\left(f(x + 1/n) - f(x)\right)$$

In topology $\quad S^n \longrightarrow *$

$$F'(X) = \varinjlim_n \left(S^{-n} \wedge \operatorname{hfiber}\left(F(X \vee S^n) \to F(X) \right) \right)$$

e.g.

$$\begin{tikzcd} G_n \arrow[r] & F(X \vee S^n) \arrow[r] \arrow[d] & F(X) \\ & F(X \vee S^{n+1}) & \end{tikzcd}$$



Get $\quad G_n \longrightarrow \operatorname{hpull}\left(\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & G_{n+1} \end{tikzcd} \right) = \Omega G_{n+1}$

e.g.

$f(x) = x^n$ $\qquad$ $F(X) = \Sigma_i^\infty X^{\wedge n}$

$F'(X) = \bigvee_n \Sigma_i^\infty X^{\wedge n-1}$

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f analytic at 0 if

$$f(x) = \varinjlim T_n f(x)$$

$$\frac{f^{(n)}(0)}{n!} x^n = T_n f(x) - T_{n-1} f(a)$$

for $|x| < R$ $\qquad$ R = radius of convergence

F is analytic at $X = *$ if

$$F(X) = \varinjlim T_n F(X)$$

when

$$\underset{h\Sigma_n}{F^{(n)}_{(*)} \wedge \Sigma_i^\infty X^{\wedge n}} \simeq \mathrm{hofiber}\left(T_n F(X) \to T_{n-1} F(X)\right)$$

for X $\underbrace{(\rho+1)}_{R\,=\text{ radius of convergence}}$-connected

i.e. $\pi_{\leq \rho} X \cong 0$

Calculus $f'(x) = g'(x)$

$$\Longrightarrow \quad f'(x) - f(0) = g'(x) - g(0)$$

(uses Mean value thm)

Lem (Goodwillie) F, G p-analytic

$$F \longrightarrow G$$

$$F' \xrightarrow[\simeq]{} G'$$

$\Longrightarrow$

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & G(X) \arrow[d] \\ F(0) \arrow[r] & G(0) \end{tikzcd}$$

is pullback for X $(p+1)$-connected

(pf) Taylor towers only differ by constant term.

Idea: apply this to

$$K \xrightarrow[tr]{} TC$$

Thm (Dundas - McCarthy)

$$K^s(R,M) := \frac{d}{dX}\Big|_{X=0} K(HR \ltimes HM \wedge X)$$

$$\simeq THH(R,M)$$

Thm (Hesselholt)

$$TC^s(R,M) := \frac{d}{dX}\Big|_{X=0} TC(HR \ltimes HM \wedge X)$$

$$TC^s(R,M)^\wedge \simeq THH(R,M)^\wedge$$

Thm

$$\begin{tikzcd}
\frac{d}{dX} K(HR \ltimes HM \wedge X)^\wedge \arrow[rr, "\simeq", "trc"'] \arrow[dr, "\simeq"] & & \frac{d}{dX} TC(HR \ltimes HM \wedge X)^\wedge \arrow[dl, "\simeq"'] \\
& THH(HR \ltimes HM \wedge X, HM)^\wedge &
\end{tikzcd}$$

Thm both K and TC are $E(1)$-analytic

Thm (McCarthy) [Relative thm]

If $A \to B$ has nilpotent kernel,

then is a htpy pullback

$$\begin{tikzcd} K(A)^\wedge \arrow[r] \arrow[d] & TC(A)^\wedge \arrow[d] \\ K(B)^\wedge \arrow[r] & TC(B)^\wedge \end{tikzcd}$$

(pf) Step 1

Form simplicial resolutions of A, B by free algebras to reduce to the case

$$R \ltimes M \longrightarrow R$$

Step 2

$$\begin{tikzcd} K(R \ltimes M)^\wedge \arrow[r] \arrow[d] & TC(R \ltimes M)^\wedge \arrow[d] \\ K(R)^\wedge \arrow[r] & TC(R)^\wedge \end{tikzcd}$$

$\left(X = S^0 \text{ is } -1 \text{ connected} \right)$

$X = *$

Where we are going:

Absolute thm

Hesselholt - Madsen k = perfect fld

For finite $W(k)$-alg's A

$$\mathrm{trc} : K_i(A, \mathbb{Z}_p) \xrightarrow{\simeq} TC_i(A, \mathbb{Z}_p) \quad i \geq 0$$

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e.g.

$$TR(\mathbb{F}_p) = W(k)$$

$$\Rightarrow \quad TC(\mathbb{F}_p) = H^*\left(TR(\mathbb{F}_p) \xrightarrow[1-F]{} TR(\mathbb{F}_p)\right)$$

$$TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0 \\ \Sigma^{-1}\mathbb{Z}_p, & * = -1 \end{cases}$$

$$\Sigma^{-2} H\mathbb{Z}_p \longrightarrow K(\mathbb{F}_p)^\wedge_p \longrightarrow TC(\mathbb{F}_p)^\wedge_p$$

$$\Leftrightarrow \quad \Sigma^{-2} H\mathbb{Z}_p \longrightarrow K(\mathbb{Z}/p^n)^\wedge_p \longrightarrow TC(\mathbb{Z}/p^n)^\wedge_p$$

$\Rightarrow$ iso on π_i $i \geq 0$.

analyze $\lim\limits_{\leftarrow n}$

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