Barwick - Curves in K-thy, the return

Note Title 4/17/2009

Recap:

Idea: $K(-) : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$

is a Green functor

Green fncts$/S$ $\overset{\Omega C_*}{\rightsquigarrow}$ Grad Witt Complexes

Rmk: $X \xrightarrow{G} S$ étale Galois $\Rightarrow$ get genuine G-spectrum $K(X)$

of spectral cat

finite $\mathbb{Z}$ sets

$$\begin{tikzcd} \mathcal{B}^{\mathbb{Z}} := \mathcal{B}^{op}_{\mathbb{Z}} \\ \mathcal{B}^{\Phi} \arrow[u, hook] \end{tikzcd}$$

Spectral category

finite discrete $\mathbb{Z}$ sets

Φ = poset of positive integers by divisibility

$$M : \pi_0 \mathcal{B}^{\Phi} \longrightarrow Ab$$

$W_n \in Ab \qquad n \geq 1$

$m | n \qquad F_{m|n} : W_n \longrightarrow W_m$

$m | n \qquad V_{m|n} : W_m \longrightarrow W_n$

S.t. - - - - -

$$F_{l|n} \circ V_{m|n} = \left(\frac{n}{\mathrm{lcm}(l,m)}\right) V_{\gcd(l,m)|l} \circ F_{\gcd(l,m)|m}$$

$l | m | n$

If M is a green functor

- $F_{m|n}(xy) = F_{m|n}(x)\, F_{m|n}(y)$
- $V_{m|n}(F_{m|n}(x) \cdot z) = x\, V_{m|n}(z)$

Def: R = ring

① Grand Witt cx /R is

a comm Green functor
(E_∞)

$$A : \mathcal{B}^{\mathbb{Z}} \longrightarrow Sp_{\geq 0}$$ spectral functor

w/ connection, and an isomorphism

$$\pi_0 A \longrightarrow W(R)$$

of Green functors $\pi_0 \mathcal{B}^{\mathbb{Z}} \to Ab$

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② The htpy initial object of ctgy of Grand Witt Cx's is called the Grand de Rham-Witt cx.

$$\mathbb{W}\Omega_R$$

$\pi_* \mathbb{W}\Omega_R$ = big de Rham witt cx

Connection: = bunch of functions $A_{\langle n \rangle}$

together w/ "restriction maps":

$\{W_{\langle n \rangle}(R)\}_n$ w/ Restriction

gives $W(R)$

Claim: $\exists$ cone automatically

from the fact that

$W\Omega_R$ is spectral valued.

Thm: (Bloch - Hesselholt) R = regular ring $/S$

$$\Omega G_* K(R) \simeq W\Omega_R$$

If $R = \mathbb{Z}_p$-alg both sides agree

$$\prod_{(p,k)=1} p\text{-typical}\left(\Omega C_* K(R)\right) \simeq \prod_{(p,k)=1} p\text{-typical}\left(\mathbb{W}^{(p)} \Omega_R\right)$$

equivalent factors

free ab for X = regular/S

Make sense of $C_* K(R)$

Rmk:

$$A : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$

given X/S

get $C_* A(X) : \mathcal{B}^{\mathbb{Z}} \longrightarrow Sp_{\geq 0}$

In fact

$$\mathcal{B}_{/S} \longrightarrow \mathrm{Mack}(\mathcal{B}^{\mathbb{Z}}, Sp_{\geq 0})$$

Def: A connection on a mackey functor

$$M : \mathcal{B}^{\mathbb{Z}} \longrightarrow Sp_{\geq 0}$$

is a compatible sequence of equivalences of Mackey functors

$$\rho : \beta_{n,!} M \xrightarrow{\simeq} M$$

$$\beta_n : \mathcal{B}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$$
$$X \longmapsto n_! X$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "n"] \arrow[d, "X"'] & \mathbb{Z} \arrow[dl, dashed, "n_! X"] \\ \text{Sets} & \end{tikzcd}$$

left kan extension

"Think induction"

$$\mathbb{Z}/m\mathbb{Z} \longmapsto \mathbb{Z}/nm\mathbb{Z}$$

$\beta_{n,!} : \mathrm{Mack}^{\mathbb{Z}} \longrightarrow \mathrm{Mack}^{\mathbb{Z}}$

enriched left kan extension

$$\Phi \longrightarrow \{Sp_{\geq 0}\text{-categories}\}$$

$$m \longmapsto \mathrm{Mack}^{\mathbb{Z}}$$

$$m/n \longmapsto \beta_{n/m,!}$$

$\underbrace{\text{Formal construction of cat of mack functors w/ correction}}_{\text{Compatible}}$

$$\begin{array}{ccc} \mathrm{Ob}\,\Phi & \longrightarrow & \Phi \\ \downarrow & & \downarrow \\ * & \longrightarrow & C\Phi \end{array}$$

$$\beta : C\Phi \longrightarrow \{\text{Spatial cats}\}$$

$$\mathrm{holim}\,\beta =: \mathrm{Mack}^{\mathbb{Z}}_{\mathrm{conn}}$$

Rmk: can form a semi-model cat of Mackey functors w/ connections, can do no better.

[**Conj:** This is same as cyclotomic spectra?]

$m \mid n$

$$\begin{tikzcd} M_n \arrow[r] \arrow[dr, "R_{m/n}"'] & \left(\beta_{n/m\,!} M\right)_m \arrow[d, "\rho_{n/m}"] \\ & M_m \end{tikzcd}$$

"restriction map"

Claim: There are restriction maps

A Green functor w/ connection,

$\rho : \mathcal{B}_{n,!} M \longrightarrow M$

is an equivalence of Green functors

<u>Rmk:</u> <u>cannot</u> extend this to $\mathcal{B}_s$.

due to finiteness <u>issues</u>.

Curves Functor:

$\mathcal{B}_{/S}$:

$$\Theta_n : \mathcal{B}_{/S} \longrightarrow \mathcal{B}_{/S}$$
$$X \longmapsto X[t]/t^n$$

$$p_X : X[t]/t^n \longrightarrow X$$

$$A : \mathcal{B}_{/S} \longrightarrow Sp_{\geq 0}$$

$$T^n A := \Theta_n^* A$$

$$T^n A \xrightarrow[p]{} A$$

p gives such a morphism of Green functors

$$C_n A \longrightarrow T^n A \xrightarrow[p]{} A$$

<u>$m|n$:</u>

$$\phi_{m|n,X} : X[u]/u^n \longrightarrow X[t]/t^m$$

$$u^{n/m} \longleftarrow\!\!\shortmid \; t$$

$$\phi_{m|n} : \Theta^n \longrightarrow \Theta^m$$

get:

$$\phi^*_{m|n} : T^m A \longrightarrow T^n A$$

map of green functors

$$(\phi_{m|n})_* : T^n A \longrightarrow T^m A$$

morph of $T^n A$-modules

$\phi^*_{m|n}$:

$$\begin{tikzcd} T^m A \arrow[r] \arrow[d] & T^n A \arrow[d] \\ A \arrow[r, equal] & A \end{tikzcd}$$

$(\phi_{m|n})_*$:

$$\begin{tikzcd} T^n A \arrow[r] \arrow[d, "p_*"'] & T^m A \arrow[drr, "p_*"] & & \\ A \arrow[r, "p^*"'] & T^{n/m} A \arrow[rr, "p_*"'] & & A \end{tikzcd}$$

recently $p : X[t]/_{t^n} \longrightarrow X$

$$
\begin{tikzcd}
C_m A \arrow[r, "V_{m|n}"'] \arrow[d] & C_n A \arrow[d] \\
T^m A \arrow[r, "\Phi^*_{m/n}"'] \arrow[dr] & T^n A \arrow[d] \\
& A
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_n A \arrow[r, "F_{m|n}"'] \arrow[d] & C_m A \arrow[d] \\
T^n A \arrow[r] \arrow[d] & T^m A \arrow[d] \\
A \arrow[r, "p_* p^*"'] & A
\end{tikzcd}
$$

Important! For $A = K$ this is mult by n/m

Rmk $\Omega C_* A \longleftrightarrow C_* A$

$THH \longleftrightarrow \mathrm{Der}$

$$\pi_0 \Omega C_m K(R) \longrightarrow \mathbb{W}_m(R)$$

truncation set

1 2 3 4 ... m

but $\mathbb{W}_m(R) \neq \mathbb{W}_{\langle m \rangle}(R)$

truncation set

$d \mid m$

Point

$m \mid n$

$$C_{\langle n \rangle} A(R) \xrightarrow{R_{m|n}} C_{\langle m \rangle} A(R)$$

Cones from:

C_m's are compatible

by restriction of jet

maps

$$C_{\langle m \rangle} \simeq \underset{d|m}{\overset{\text{Rest maps}}{\mathrm{holim}}}\ C_m$$

Cor: easily use functoriality

to make $\Omega C_{>} A$

a green functor.