<u>Intermediate Topology/Geom pset 4</u>

Assigned: 9/23/16,

"Due": 9/30/16

<u>(do 1-3, and either 4a or 4b)</u>

1) Suppose $\alpha: S^{4n-1} \to S^{2n}$ is a map. Let $X = S^{2n} \cup_\alpha e^{4n}$ be the resulting 2-cell complex. Then $H^2(X) \cong H^4(X) \cong \mathbb{Z}$. Let x be the degree 2 generator, and y the degree 4 generator. We have
$$x^2 = ky$$
for k an integer. The Hopf invariant is defined by
$$HI(\alpha) = |k|$$
(e.g. the Hopf fibration has Hopf invariant 1). Use steenrod operations to show that if α has Hopf invariant 1, then all of the suspensions
$$\Sigma^k \alpha: S^{4n-1+k} \to S^{2n+k}$$
are not null homotopic. (this implies that Hopf invariant 1 maps give non-trivial elements of the stable homotopy groups of spheres.)

2) Show that the action of the steenrod algebra on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ is given by the formula
$$Sq^i(x^j) = \binom{j}{i} x^{i+j}$$

3) Deduce from the above that if there is a map
$$S^k \to \mathbb{R}P^\infty$$
which is an isomorphism on $H^k(-; \mathbb{F}_2)$, then $n = 2^i - 1$.

(Thus the only n for which the Hurewicz homomorphism $\pi_{2n-1}(\mathbb{R}P^\infty) \to H_{2n-1}(\mathbb{R}P^\infty)$ could be surjective are $n = 2^{i-1}$. These turn out to be the same dimensions n where you can have a Hopf invariant one map $S^{4n-1} \to S^{2n}$.)

4a) $\mathbb{C}P^\infty$ is an H-space, and its homology and cohomology are Hopf algebras. Consider the Hopf algebra $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with coproduct determined by:
$$\psi(x) = x \otimes 1 + 1 \otimes x$$
Show that the dual Hopf algebra
$$H_*(\mathbb{Z}[x])^* = Hom(\mathbb{Z}[x], \mathbb{Z})$$
is isomorphic to the divided power algebra $\Gamma[x]$. That is to say, if γ_i is dual to x^i, then
$$\gamma_i \cdot \gamma_j = \binom{i+j}{i} \gamma_{i+j}$$

4b) Prove the following "baby version" of Milnor's theorem. Let A(1) be the subalgebra of the mod 2 steenrod algebra generated by Sq^1 and Sq^2 . Explicitly,

$$A(1) = \mathbb{F}_2\langle Sq^1, Sq^2\rangle/(Sq^1Sq^1 = 1, Sq^1Sq^2Sq^1 = Sq^2Sq^2)$$

Prove the dual of A(1) is given by

$$A(1)_* = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$$

With coproduct:

$$\psi(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$$

$$\psi(\xi_2) = \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2$$