7-Orientation thy, duality

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Note: E = ring spectrum $\Rightarrow$ E_*X, E^*X are E_*-modules

Thom cx $X^V := D(V)/S(V)$

$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd}$$

$$(X \times Y)^{V \times W} \cong X^V \wedge Y^W$$

$$X^{V \oplus \mathbb{R}^n} \cong \Sigma^n X^V \qquad (X^{\mathbb{R}^n} = \Sigma^n X_+)$$

cells of X^V = shifts of cells of X_+

$$X^{[n-1]} \to X^{[n]} \to \bigvee_i S^n$$

$$\begin{tikzcd} V^{[n]} \arrow[r] \arrow[d] & V \arrow[d] \\ X^{[n]} \arrow[r] & X \end{tikzcd}$$

$$(X^{[n-1]})^{V^{[n-1]}} \to (X^{[n]})^{V^{[n]}} \to \bigvee_i S^{n+k}$$

$\leftarrow$ use fact that bundle trivializes over cells.

Orientation thy

$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd}$$

v.b. rank n

X^V = thom complex $= D(V)/S(V)$

E = comm. ring spectrum

Def: V is E-orientable if

$\exists\ [V] \in \tilde{E}^*(X^V) \cong E^*(D(V), S(V)) \cong E^*(V, V-0)$ $\quad \pi_{n-*}E$ $\cong$

(typically $[V] \in \tilde{E}^n(X^V)$)

Thom class

s.t. $\forall x$ $\quad \tilde{E}^*(X^V) \to \tilde{E}^*(D(V)_x/S(V)_x)$

$$[V] \longmapsto [V]_x$$

$[V]_x$ generates $\tilde{E}^*(D(V)_x/S(V)_x)$ as an E_*-module.

Relationship to classical orientations

V orientable $\Leftrightarrow$ $H\mathbb{Z}$-orientable.

Local orientations: elts of $\tilde{H}^*(U^V) \cong \tilde{H}^*(U_+ \wedge S^n) \cong H^*(U) \otimes \tilde{H}^*(S^n)$ $\quad 1 \otimes \pm \iota_n$

patch together to give global orientations

$$[V] \in H^n(X^V)$$

Thom Diagonal

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \times V \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

Thom Diagonal

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \times V \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

gives $\quad X^V \longrightarrow (X \times X)^{0 \times V} = X_+ \wedge X^V$

e.g. $\quad E^*(X) \otimes_{E_*} \tilde{E}^*(X^V) \to \tilde{E}^*(X^V)$

Thom isomorphism thm $\quad$ If V is E-orientable (rank k)

$$\begin{aligned} E^*(X) &\xrightarrow{\cong} \tilde{E}^{*+d}(X^V) \\ x &\longmapsto x[V] \end{aligned} \qquad [V] \in \tilde{E}^d(X^V)$$

This follows from Geometric Thom iso

$$X^V \xrightarrow{[V]} \Sigma^d E$$

$$\begin{tikzcd} E \wedge X^V \arrow[r] \arrow[rrr, bend right, "\Phi"'] & E \wedge X_+ \wedge X^V \arrow[r] & \Sigma^d E \wedge E \wedge X_+ \arrow[r] & \Sigma^d E \wedge X_+ \end{tikzcd}$$

$$\begin{tikzcd} E \wedge (X^{(n-1)})^{V^{n-1}} \arrow[r] \arrow[d, "\simeq"] & E \wedge (X^{(n)})^{V^n} \arrow[r] \arrow[d] & E \wedge \bigvee S^{n+d} \arrow[d, "\simeq"] \\ \Sigma^d E \wedge X^{(n-1)}_+ \arrow[r] & \Sigma^d E \wedge X^{(n)}_+ \arrow[r] & \Sigma^d E \wedge \bigvee S^n \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^{-d} F(X_+, E) \arrow[r] \arrow[rrr, bend right, "\Phi \; \simeq"'] & F(X^V, E) \wedge F(X_+, E) \arrow[r] & F(X^V \wedge X_+, E \wedge E) \arrow[r] & F(X^V, E) \end{tikzcd}$$

Rmk: There is always an iso $\quad$ (V non-orientable)

$$H^*(X; \mathbb{Z}^\sigma) \cong \tilde{H}^{*+d}(X^V)$$

$$\mathbb{Z}^\sigma = H^d(V_x, V_x - 0) \cong \mathbb{Z} \circlearrowleft \pi_1(X)$$

Spanier–Whitehead Duality

$$X \in Sp \qquad DX := F(X,S)$$

$$[Z, DX] \simeq [Z \wedge X, S],$$

$$\left(\begin{array}{l} A \to B \to C \quad \text{cofiber} \\ \Rightarrow A \wedge Z \to B \wedge Z \to C \wedge Z \quad \text{cofiber} \\ F(-,Z),\ F(Z,-) \\ \text{also cofiber} \end{array} \right)$$

$$F(A,B) \wedge C \longrightarrow F(A, B \wedge C)$$

equivalence if A or C finite

Note $D(S^n) = S^{-n}$, D preserves cofiber sequences

$$D(\bigvee_i) = \prod_i$$

$$X^{[k]}/X^{[k-1]} \longrightarrow X/X^{[k-1]} \longrightarrow X/X^{[k]}$$

$$X^{[k]}/X^{[k-1]} \simeq \bigvee_i S^k$$

$$\Rightarrow \quad \prod S^{-k} \leftarrow D(X/X^{[k-1]}) \leftarrow D(X/X^{[k]})$$

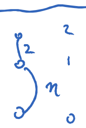
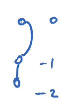
Consequence: X finite $\Rightarrow$ DX finite

$$X \to D(DX) \qquad \text{adjoint to} \qquad X \wedge DX \xrightarrow[ev]{} S$$

$$\updownarrow$$

$$DX \xrightarrow[Id]{} DX$$

$$X \text{ finite} \Rightarrow X \xrightarrow{\simeq} D(DX)$$

e.g. $X =$  $DX =$ 

$$E \wedge DX \simeq E \wedge F(X,S) = F(X,E)$$

X finite

$$\Rightarrow \boxed{E_* DX \simeq E^{-*}(X)}$$

Poincaré duality:

P-T:

$$M \hookrightarrow \mathbb{R}^N \qquad TM \oplus \nu = \mathbb{R}^N$$

$$\Rightarrow \nu = -TM + \mathbb{R}^N$$

$$=$$

$$M^{-TM} \simeq \Sigma^{-N} M^{\nu}$$

$$S^N \longrightarrow M^{\nu}$$

$$\Rightarrow \quad S \xrightarrow[{[M]}]{} M^{-TM} \quad \text{in SHC}$$

e.g. $H_0 M^{-TM} \simeq H_d(M;\mathbb{Z}^{\omega})$

$\ni [M] \qquad \ni [M] \leftarrow$ fund class

$$S \longrightarrow M^{-TM} \longrightarrow M_+ \wedge M^{-TM}$$

gives $DM_+ \longrightarrow M^{-TM}$

Thm (Atiyah duality) This map is an equivalence.

(Pf)

$$\begin{array}{ccc} H_{-*} DM_+ & \longrightarrow & H_* M^{-TM} \\ \wr\wr & & \wr\wr \\ H^*(M) & & H_{d-*}(M;\mathbb{Z}^{\omega}) \end{array}$$

(Classical P.D. says this is a H_*-iso.)

$\Rightarrow$ this is an iso.

H_*-Whitehead $\qquad \square$

Consequence: Generalized Poincare Duality

Suppose M is E-orientable

$$\begin{array}{ccc} E_{-*}(DM_+) & \xrightarrow[\cong]{\text{Atiyah}} & E_{-*}(M^{-TM}) \\ \wr\wr \text{ S-W} & & \wr\wr \text{ Thom} \\ E^*(M) & \xrightarrow[\cong]{\text{Poincare}} & E_{d-*}(M) \end{array}$$
