3 - Fiber and cofiber sequences, smash and function

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Ω, Σ two ways

(1) $\Sigma_1 X = X \wedge S^1$

(2) $\Omega X = X^{S^1}$

$X[1] := (X_1, X_2, \ldots)$

$X[-1] := (*, X_0, X_1, \ldots)$

(At least if X is CW) there are weak maps (π_*^s iso)

$\Sigma_1 X \xrightarrow{w} X[1]$

$$\begin{array}{ccc} \Sigma_1 \Sigma_1 X_0 & \to & \Sigma_1 X_{i+1} \\ \downarrow & & \downarrow \\ \Sigma_1 X_{i+1} & \to & X_{i+2} \end{array}$$

Similarly $X[-1] \xrightarrow{w} \Omega X$

$$\begin{array}{ccc} X & \longrightarrow & X[1][-1] \\ \downarrow & & \downarrow w \\ \Omega\Sigma X & \xrightarrow{w} & \Omega X[1] \end{array}$$

commutes

$X[-1][1] \rightleftarrows X$

Warning: Does not commute

$\Rightarrow$ $X \to \Omega \Sigma_1 X$ is a st. equiv.

Similarly $\Sigma_1 \Omega X \to X$ is a st. equiv.

Cor: $[X, Y]^s \cong [\Sigma_1 X, \Sigma_1 Y]^s = [\Omega X, \Omega Y]^s$

$SW \longrightarrow SHC \longrightarrow$ Coh. Thy's

(already know this functor)

$(X, n) \longmapsto (X, n)$

$$[(X, n), (Y, m)]_{Sp} = [\Sigma^i X, \Sigma^i Y]_{i = \min(n, m)}$$

$$\Rightarrow \quad SW((X,n),(Y,m)) \longrightarrow \varinjlim_i [\Sigma^i(X,n), \Sigma^i(Y,m)]_{Sp}$$

$$\downarrow$$

$$[(X,n),(Y,m)]^s \cong \varinjlim_i [\Sigma^i(X,n), \Sigma^i(Y,m)]^s$$

Prop:

$$[X,Y]^s = [\tilde{X}, \omega Y]_{Sp}$$

(pf)

$$[\widetilde{\omega X}, \widetilde{\omega Y}]_{Sp} \xrightarrow{\cong} [\widetilde{\omega X}, \omega Y]_{Sp}$$

$$\uparrow \cong$$

$$[\tilde{X}, \omega Y]_{Sp}$$

Exercise

$f: X \to Y$ levelwise equiv, Z = CW spectrum $\Longrightarrow$ $f_*: [Z, X]_{Sp} \to [Z, Y]_{Sp}$ is an iso

(use $[(S^n, i), X]_{Sp} = \pi_n(X_i)$)

Exercise

- $f : X \to Y$ stable equiv between CW spectra, Z Ω-spectrum $\Rightarrow$ $f^* : [Y, Z]_{Sp} \to [X, Z]_{Sp}$ is an iso

$[X, Z]_{Sp}$

$$\begin{array}{ccc} X_i & \to & Z_i \\ \downarrow & \nearrow & \\ \Omega X_{i+1} & & \end{array}$$

show $X \to \omega X$ preserves this

Q: is $X[-1] \to \Omega X$ natural?

Cohomology

$\tilde{E}^*$ a coh thy w/ spectrum E

$$\begin{array}{ccc} Ho(Top_*) & \xrightarrow{\tilde{E}^*} & Gr(Ab) \\ \downarrow & \nearrow & \\ SW & & \\ \downarrow & \nearrow_{E^*} & \\ SHC & & \end{array}$$

$E^*(X) = [X, \Sigma^* E]^s$

Rmk: Sp enriched in ab gps

[Exercise: this extends $\tilde{E}^*$]

[Exercise: lim' sequence]

$SHC \xrightarrow{\simeq} \text{Coh. thy}(Sp)$

$\Rightarrow$ Nat trans in SHC $X[-1] \to \Omega X$, $\bar{X} = X$

SHC enriched in Ab $\Rightarrow$

Cor: finite wedges = finite products

[**Exercise** $\mathcal{C}$ enriched in Ab $\Rightarrow$ finite coproducts = finite products]

$[S^{-i} \Sigma^\infty K, E]$

Cor: $\Sigma^\infty : Ho(Top_*) \rightleftarrows SHC : \Omega^\infty$ $[\Sigma^{-i}\Sigma^\infty K, E]$ =

(Spectra as ∞-loop spaces) $\tilde{E}^i(K) = [\Sigma^\infty K, \Sigma^i E]^s$

(Sphere Spectrum) From now on $[-,-] = [-,-]^s$, $\pi_* = \pi_*^s$

$X \to Y \to Cf$ ⟹ Puppe sequence

$Ff \to X \to Y$ ⟶ LES

$$\begin{array}{ccccccc} F(f) & \to & X & \to & Y & \to & \Sigma F(f) \\ \downarrow & & \| & & \| & & \downarrow \\ \Omega C(f) & \to & X & \to & Y & \to & Cf \end{array}$$

} LES of htpy grps

iso.

Function spectrum $X, Y \in Sp \rightsquigarrow F(X,Y) \in Sp$

Defining property:

$$[\Sigma^i \Sigma^\infty K, F(X,Y)] = [\Sigma X \wedge K, Y]$$

check this is coh thy.

Represent it by a spectrum.

Note: $\pi_i F(X,Y) = [\Sigma^i S, F(X,Y)] = [\Sigma^i X, Y]$

Point set models

$$[(K,i), F(X,Y)] = [K, \underline{F(X,Y)_i}]_*$$

$$= [\Sigma^{-i} X \wedge K, Y]$$

Choose Y - Ω spectrum

$$= [\Sigma^{-i} X \wedge K, Y]$$
$$= [X \wedge K, \Sigma^i Y]$$

Choose Y - Ω spectrum
X = CW spectrum.

$$\underline{\mathrm{Map}}(X, Y) \hookrightarrow \prod \underline{\mathrm{Map}}_*(\underline{X}_i, \underline{Y}_i)$$

$$\pi_0 \underline{\mathrm{Map}}(X, Y) = [X, Y]$$

[Exercise $\pi_i \underline{\mathrm{Map}}(X, Y) = [\Sigma^i X, Y]$]

$$\Omega \underline{\mathrm{Map}}(X, Y) \cong \underline{\mathrm{Map}}(X, \Omega Y) = \underline{\mathrm{Map}}(\Sigma X, Y)$$

$$\underline{\mathrm{Map}}(X, Y) \simeq \Omega \underline{\mathrm{Map}}(X[-1], Y) \simeq \Omega^2 \underline{\mathrm{Map}}(X[-2], Y) \cdots$$

Define $\underline{F(X,Y)}_i = \underline{\mathrm{Map}}(X[-i], Y)$ This makes $F(X,-)$ a functor

Smash Product:

Defining property $[X \wedge Y, Z] \cong [X, F(Y, Z)]$

($X \wedge Y$ unique if it exists)

Intuition $(K, i) \wedge (L, j) = (K \wedge L, i+j)$

More generally $\underline{X \wedge Y}_i = \underline{X}_{i_1} \wedge \underline{Y}_{i_2}$ $\quad i_1 + i_2 = i$

$$\Sigma \underline{X}_{i_1} \wedge \underline{Y}_{i_2} \xrightarrow{?} \underline{X}_{i_1} \wedge \underline{Y}_{i_2+1}$$
$$\downarrow ?$$
$$\underline{X}_{i_1+1} \wedge \underline{Y}_{i_2}$$

Def: (Boardman) $n(i)$, $m(i)$ functions of i

- $n(i) + m(i) = i$
- increasing
- $n(i) \to \infty$
  $m(i) \to \infty$

D f $X \wedge Y$ ...

Define $\underline{X} \wedge \underline{Y}_i = \underline{X}_{n(i)} \wedge \underline{Y}_{m(i)}$ w/ evident structure maps

Thm $[X \wedge Y, Z] \simeq [X, F(Y, Z)]$

$[X \wedge Y, Z] = [X, F(Y, Z)]$ "two different spectrum structures on Y"

$Y \quad \bar{Y}$

$\sigma \quad \bar{\sigma}$

$\bar{\sigma}(s, \sigma(t, y)) = \sigma(t, \bar{\sigma}(s, y))$

$Y \simeq \bar{Y}$

$\bar{\sigma} Y = \overline{\sigma Y}$

Idea: $\underline{X}_i \longrightarrow \underline{\mathrm{Map}}(Y[-i], Z) \longleftrightarrow \prod_j \underline{\mathrm{Map}}_*(\underline{Y}_{j-i}, \underline{Z}_j)$

$x \longmapsto f_i(x)_j$

Note $f(\sigma(t, y)) = \sigma(t, f(y))$

Spectrum structure $\sigma(t, f)(y) = f(\bar{\sigma}(t, y))$

So $f(\sigma(t, x))(y) = f(x)(\bar{\sigma}(t, y))$

and $f(x)(\sigma(t, y)) = \sigma(t, f(x)(y))$

$(x, y) \longmapsto f(x, y)$

$f(\sigma(t, x), \bar{y}) = f(x, \bar{\sigma}(t, \bar{y}))$

$= f(x, \overline{\sigma(t, y)})$

$= \sigma(t, f(x, y))$

Y a bimodule

$X \wedge Y$

$$\bigvee_{i+j=k} \underline{X}_i \wedge \underline{Y}_j \Big/ \sim \qquad (\sigma(x), y) \simeq (x, \bar{\sigma}(y))$$

$$\sigma(x, y) = (x, \sigma(y))$$

Exercise: handcrafted models some spectra ...

Argue there is a weak map

$$- \underset{\text{handcraft}}{\wedge} - \longrightarrow - \wedge -$$

Exercise $\rightsquigarrow$ descends to symmetric monoidal structure on Spectra. (use different handi-crafts)

Exercise $\rightsquigarrow$ $- \wedge -$, $F(-,-)$ commute w/ cofiber sequences.

Note: $[X \wedge \Sigma^\infty_\cdot K, Y] \equiv [\Sigma^\infty_\cdot K, F(X, Y)] = [X \wedge K, Y]$

$$\Rightarrow X \wedge \Sigma^\infty_\cdot K \simeq X \wedge K$$

$X \wedge S \simeq X$ $\qquad F(S, X) \simeq X$
