

10 - Universal coefficient spectral sequence

Monday, October 27, 2014 10:36 AM

Classically

$$0 \rightarrow \bigoplus_{i_1+i_2=i} \tilde{H}_{i_1}(X) \otimes \tilde{H}_{i_2}(Y) \rightarrow \tilde{H}_i(X \wedge Y) \rightarrow \bigoplus_{i_1+i_2=i-1} \text{Tor}_{\mathbb{Z}}^1(\tilde{H}_{i_1}(X), \tilde{H}_{i_2}(Y)) \rightarrow 0$$

$$0 \rightarrow H_i(X) \otimes A \rightarrow H_i(X; A) \rightarrow \text{Tor}_{\mathbb{Z}}^1(H_{i-1}(X), A) \rightarrow 0$$

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{i-1}(X), A) \rightarrow H^i(X; A) \rightarrow \text{Hom}(H_i(X), A) \rightarrow 0$$

Generalize these...

(1) Generalized homology of coefficients.

Given $A \in \text{Ab}$

$M(A)$ = Moore spectrum

$$0 \rightarrow \bigoplus_i \mathbb{Z} \rightarrow \bigoplus_i \mathbb{Z} \rightarrow A \rightarrow 0$$

form
c.fiber.

$$\bigvee_i S \rightarrow \bigvee_i S \rightarrow M(A)$$

$$\pi_* H\mathbb{Z} \wedge M(A) = H_* M(A) = \begin{cases} A, & * = 0 \\ 0, & \text{else} \end{cases}$$

e.g. $M(\mathbb{Z}/2) \simeq \Sigma \tilde{\mathbb{R}P}^2$

$$H\mathbb{Z} \wedge M(A) \simeq HA$$

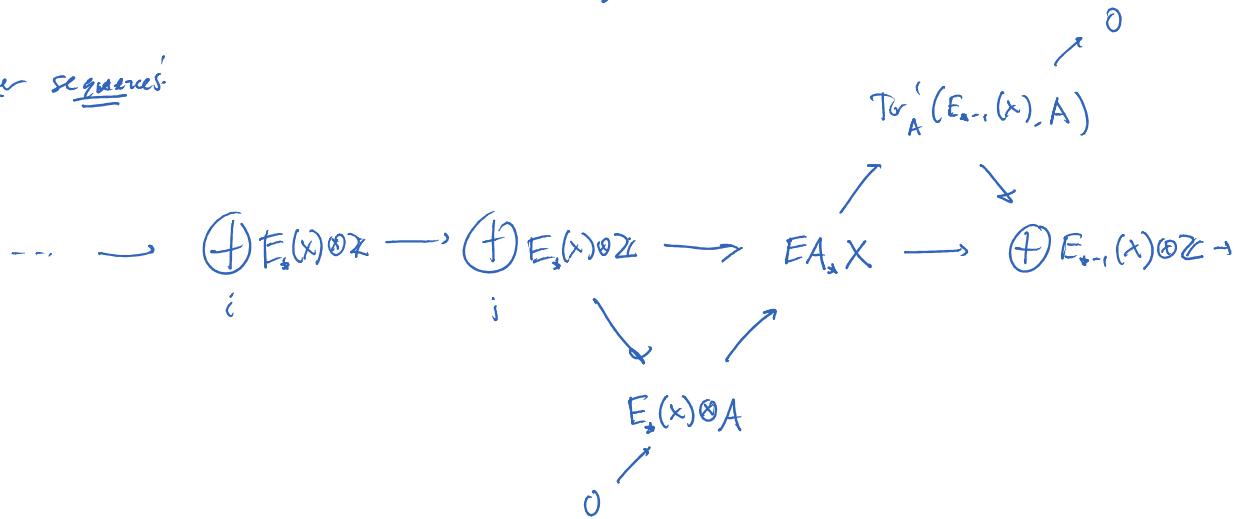
More generally, for E a spectrum, define

$$EA := E \wedge M(A)$$

$$EA_* X \simeq E_*(X; A)$$

$$EA_*X \cong E_*(X; A)$$

Cofiber sequences



Cool examples

(1) $A = \mathbb{Z}[S^{-1}]$ $S = \text{set of primes}$ $\text{Tor}_1 = 0$ (flat)

localization

Denote $XA = X(S^{-1})$

$$\pi_0(X(S^{-1})) = \pi_0(X)[S^{-1}]$$

eg. $\mathbb{Z}[1/p]$ "avoid a bad prime"

$\mathbb{Z}_{(p)}$ "focus in on a prime."

\mathbb{Q} "avoid torsion"

Arithmetic Square

Prop
the
is a
htpy pullback!

$$\begin{array}{ccc}
 X & \longrightarrow & X_{(p)} \\
 \downarrow & & \downarrow \\
 X[1/p] & \longrightarrow & X_{\mathbb{Q}}
 \end{array}$$

(pt) Just need to prove

$$\begin{array}{ccc}
 S & \longrightarrow & M(\mathbb{Z}_{(p)}) \\
 \downarrow & & \downarrow \\
 M(\mathbb{Z}[1/p]) & \longrightarrow & M(\mathbb{Q})
 \end{array}$$

is htpy pullback

(check on homology.)

Rational stable homotopy is trivial

What about cohomological one?
even intresting when $A = \mathbb{Z}$

$$E^*(X) \iff \text{Hom}_{E_*}(E_*(X), E_*) ?$$

Need E to be Ring spectrum

Similarity for Künneth:
$$\tilde{E}_*(X \wedge Y) \hookrightarrow \tilde{E}_*(X) \otimes_{E_*} \tilde{E}_*(Y)$$

General setup: $E =$ Ring spectrum
 $M = E$ -module spectrum.

Want

$$(1) \quad \text{Tor}_{s,t}^{E_*}(E_*X, M_*) \Rightarrow M_*X$$

$$(2) \quad \text{Ext}_{E_*}^{s,t}(E_*X, M_*) \Rightarrow M^*X$$

e.g. $M = E \wedge Y \quad \text{Tor}_{s,t}^{E_*}(E_*(X), E_*(Y)) \Rightarrow E_*(X \wedge Y)$

e.g. $M = EA$

$$\text{Tor}_{E_*}^{E_*}(E_*X, EA_*) \Rightarrow EA_*(X)$$

$$\text{Ext}_{E_*}^{E_*}(E_*X, EA_*) \Rightarrow EA^*(X)$$

Hypotheses

(a) E is a strictly associative Ring spectrum

$M =$ "strict E -module"

(b) $E = \varinjlim E_\alpha$, $\bullet \in E^+ E_\alpha$ is E_* -projective

$$\circ \text{Hom}_{E_*}^*(DE_2 \rightarrow M_0) \rightarrow \text{Hom}_{E_*}(E_* DE_2, M_0)$$

is an iso

$$\left(E_* DE_2 \otimes_{E_*} M_0 \xrightarrow{\cong} DE_2 \wedge M_0 \right)$$

Lemma
Prop. 11

$E_* Y$ is flat

$$E_* Y \otimes_{E_*} E_* X \rightarrow E_*(Y \wedge X) \quad \text{is an iso}$$

Case (a) In fact, on $F(N \wedge_E M)$, $F_E(N, M)$

$$X \wedge_E N \wedge M = X \wedge M$$

$$\text{Tor}_{E_*}^0(N, M) \Rightarrow \pi_0(N \wedge_E M)$$

$$\text{Ext}_{E_*}^1(N, M) \Rightarrow \pi_1 F_E(N, M)$$

$$F_E(E \wedge X, M) = F(X, M)$$

Key Lemma! \forall stft E -modules N , $\exists W \rightarrow N$

s.t. (1) W is a free E -module

$$W = \bigvee_i \Sigma^{n_i} E \quad \left(\begin{array}{l} \pi_0 N \cong \\ E_* \text{ free} \end{array} \right)$$

(2) $W_i \rightarrow N_i$ is surjective

(pf) sketch!

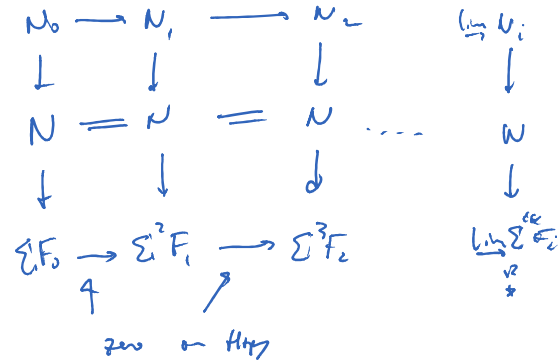
$$\bigvee_i \Sigma^{n_i} E \rightarrow N \quad \text{set of generators}$$

$$\bigvee_i \Sigma^{n_i} E = E \wedge \bigvee_i \Sigma^{n_i} \rightarrow E \wedge N \rightarrow N$$

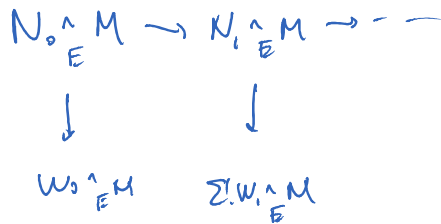
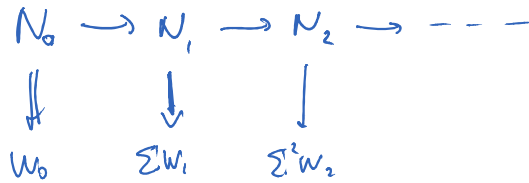
Construction of spectral sequences (es. Künneth)



Get



Get $\varinjlim N_i = N$



Get a SS

$$E_{s,t}^i = \pi_{s,t} \sum_{s \leq r \leq t} W_r \wedge_E M \Rightarrow \pi_{s,t} N \wedge_E M$$

$$\pi_{s,t} W_s \wedge_E M \cong \pi_s W_s \otimes_{E_s} M$$

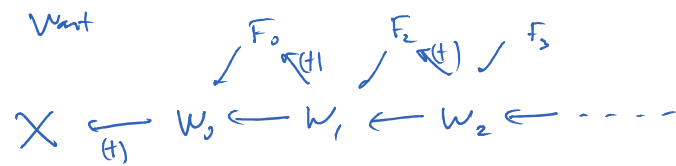
$$E_{s,t}^i = \text{Tor}_{s,t}^{E_*} (N_i, M_*)$$

Case (b)

Want

$$/ F_0 \wedge_{\mathbb{Z}(1)} / F_2 \wedge_{\mathbb{Z}(1)} / F_3$$

Case (b)



s.t. (1) $E_n W_i$ is E_n -projective

(2) $E_n \langle \theta \rangle$ are surjective

Lemma: Given X_i , get $W \rightarrow X$ s.t. $E_n W \rightarrow E_n X$ epi, $E_n W$ proj.

$$\bigvee_i S^{n_i} \rightarrow E \wedge X$$

$$\begin{array}{ccc}
 S^{n_i} & \rightarrow & E \wedge X \\
 & \searrow & \nearrow \\
 & & E_2 \wedge X
 \end{array}$$



$$S^{\infty} \wedge DE_2 \rightarrow X$$