10 - Universal coefficient spectral sequence

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Classically

$$0 \to \bigoplus_{i_1+i_2=i} \tilde{H}_{i_1}(X)\otimes \tilde{H}_{i_2}(Y) \to \tilde{H}_i(X\wedge Y) \to \bigoplus_{i_1+i_2=i-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_{i_1}(X), H_{i_2}(Y)) \to 0$$

$$0 \to H_i(X)\otimes A \to H_i(X;A) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{i-1}(X), A) \to 0$$

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X), A) \to H^i(X;A) \to \mathrm{Hom}(H_i(X), A) \to 0$$

Generalize these...

(1) Generalized homology w/ coefficients.

Given $A \in Ab$

$M(A)$ = Moore spectrum

$$0 \to \bigoplus_i \mathbb{Z} \to \bigoplus_j \mathbb{Z} \to A \to 0$$

form cofiber.

$$\bigvee_i S \to \bigvee_j S \to M(A)$$

$$\pi_* H\mathbb{Z}\wedge M(A) \simeq H_* M(A) = \begin{cases} A, & * = 0 \\ 0, & o/w \end{cases}$$

e.g. $M(\mathbb{Z}/2) \simeq \Sigma^{-1}\Sigma^{\infty}\mathbb{R}P^2$

$$H\mathbb{Z}\wedge M(A) \simeq HA$$

More generally, for E a spectrum, Define

$$EA := E\wedge M(A)$$

" $EA_* X = E_*(X;A)$ "

0

"$EA_*X \cong E_*(X;A)$"

Cofiber sequences:

$$\begin{tikzcd}
 & & & & & & 0 & \\
 & & & & & \mathrm{Tor}_A^1(E_{*-1}(X), A) \arrow[ur] \arrow[dr] & & \\
\cdots \arrow[r] & \bigoplus_i E_*(X)\otimes\mathbb{Z} \arrow[r] & \bigoplus_j E_*(X)\otimes\mathbb{Z} \arrow[rr] \arrow[dr] & & EA_*X \arrow[rr] \arrow[ur] & & \bigoplus E_{*-1}(X)\otimes\mathbb{Z} \arrow[r] & {} \\
 & & & E_*(X)\otimes A \arrow[ur] & & & & \\
 & & 0 \arrow[ur] & & & & &
\end{tikzcd}$$

Cool examples

(1) $A = \mathbb{Z}[S^{-1}]$ S = set of primes $\mathrm{Tor}_1 = 0$ (flat)

localization

Denote $XA = X[S^{-1}]$

$$\pi_*(X[S^{-1}]) = \pi_*(X)[S^{-1}]$$

e.g. $\mathbb{Z}[1/p]$ "avoid a bad prime"

$\mathbb{Z}_{(p)}$ "focus in on a prime."

$\mathbb{Q}$ "avoid torsion"

Arithmetic Square

Prop there is a htpy pullback!

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_{(p)} \arrow[d] \\
X[1/p] \arrow[r] & X_{\mathbb{Q}}
\end{tikzcd}$$

(pf) Just need to prove

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & M(\mathbb{Z}_{(p)}) \arrow[d] \\
M(\mathbb{Z}[1/p]) \arrow[r] & M(\mathbb{Q})
\end{tikzcd}$$

is htpy pullback

(check on homology.)

Rational stable homotopy is trivial

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Thm! $\pi_*(X)\otimes\mathbb{Q} \longrightarrow H_*(X;\mathbb{Q})$ is an iso!

Input: need to know $\pi_*^s \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & *=0 \\ 0, & o/w. \end{cases}$

$$\left(\text{Serre: } \pi_*(S^n)\otimes\mathbb{Q} = \begin{cases} \mathbb{Q}, & *=n \\ \mathbb{Q}, & *=2n-1, \ n \text{ even} \\ 0, & o/w \end{cases} \right)$$

$\Rightarrow$ $M\mathbb{Q} \xrightarrow[\simeq]{} H\mathbb{Z}\wedge M\mathbb{Q} \simeq H\mathbb{Q}$

$X\wedge M\mathbb{Q} \longrightarrow X\wedge H\mathbb{Q} = H\mathbb{Q}\wedge X$ is an equiv.

apply π_* . . .

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(2) $KU[1/2] \simeq KO[1/2] \vee \Sigma^2 KO[1/2]$

(Pf) $\eta \in \pi_1^s \cong \mathbb{Z}/2$ $\qquad C\eta = \Sigma^{-2}\Sigma^\infty \mathbb{CP}^2$

$$\begin{tikzcd} \pi_3(S^2)=\mathbb{Z} \arrow[d] \\ \pi_4(S^3)=\mathbb{Z}/2 \end{tikzcd}$$

$KO\wedge C\eta \simeq KU$

$$\begin{tikzcd} S^1[1/2] \arrow[r, "*"'] & S^0[1/2] \arrow[r] & C\eta[1/2] \arrow[r] & S^2[1/2] \end{tikzcd}$$

$\Rightarrow$ $C\eta[1/2] \simeq S[1/2] \vee \Sigma^2 S[1/2]$

$\Rightarrow$ $KO\wedge C\eta[1/2] = KU[1/2]$

$\quad \wr\wr$

$KO[1/2] \vee \Sigma^2 KO[1/2]$

$\pi_2 KO\mathbb{Z}/2 \cong \mathbb{Z}/4$

example of Univ. coef sequence not splitting!

What about cohomological one?

What about cohomological one? $E^*(X) \hookrightarrow \mathrm{Hom}_{E_*}(E_*(X), E_*)$?

even interesting when $A = \mathbb{Z}$

Need E to be Ring spectrum

Similarly for Künneth: $\tilde{E}_*(X \wedge Y) \hookleftarrow \tilde{E}_*(X) \otimes_{E_*} \tilde{E}_*(Y)$

General setup: E = Ring spectrum

M = E-module spectrum.

Want

(1) $\mathrm{Tor}^{E_*}_{s,t}(E_*X, M_*) \Rightarrow M_*X$

(2) $\mathrm{Ext}^{s,t}_{E_*}(E_*X, M_*) \Rightarrow M^*X$

e.g. $M = E \wedge Y$ $\qquad \mathrm{Tor}^{E_*}_{*,*}(E_*(X), E_*(Y)) \Rightarrow E_*(X \wedge Y)$

e.g. $M = EA$

$\mathrm{Tor}^{E_*}(E_*X, EA_*) \Rightarrow EA_*(X)$

$\mathrm{Ext}_{E_*}(E_*X, EA_*) \Rightarrow EA^*(X)$

Hypotheses (a) E is a strictly associative Ring spectrum

M = "strict E-module"

(b) $E = \varinjlim E_\alpha$, • $E^* E_\alpha$ is E_*-projective

- $M^* DE_\alpha \longrightarrow \mathrm{Hom}_{E_*}(E_* DE_\alpha, M_*)$ is an iso

$\Big(E_* DE_\alpha \otimes_{E_*} M_* \longrightarrow DE_\alpha \wedge M$ is an iso $\Big)$

Lemma Suppose $E_* Y$ is flat

$E_* Y \otimes_{E_*} E_* X \longrightarrow E_*(Y \wedge X)$ is an iso

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Case (a) In fact, can form $N \wedge_E M$, $F_E(N, M)$

$X \wedge E \wedge_E M \simeq X \wedge M$ $\qquad$ $\mathrm{Tor}^{E_*}(N_*, M_*) \Rightarrow \pi_*(N \wedge_E M)$

$F_E(E \wedge X, M) \simeq F(X, M)$ $\qquad$ $\mathrm{Ext}_{E_*}(N_*, M_*) \Rightarrow \pi_* F_E(N, M)$

Key lemma! $\forall$ strict E-modules N, $\exists$ $W \to N$

s.t. (1) W is a free E-module

$W \simeq \bigvee \Sigma^{(-)} E$ $\quad$ ($\pi_* N$ is E_* free)

(2) $W_* \to N_*$ is surjective

(pf) trivial:

$$\bigvee_i S^{n_i} \longrightarrow N \qquad \text{set of generators}$$

$$\bigvee_i \Sigma^{n_i} E \simeq E \wedge \bigvee_i S^{n_i} \longrightarrow E \wedge N \longrightarrow N$$

Construction of spectral sequences (eg. Künneth)

$$\begin{tikzcd}
 & & & F_0 \arrow[dl] & & F_1 \arrow[dl] & & \\
N & & W_0 \arrow[ll, "(t)"'] & & W_1 \arrow[ll] \arrow[ul, "(t)"'] & & W_2 \arrow[ll] \arrow[ul, "(t)"'] & \cdots \arrow[l]
\end{tikzcd}$$

Get

$$\begin{tikzcd}
N_0 \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[d] & \varinjlim N_i \arrow[d] \\
N \arrow[r, equal] & N \arrow[r, equal] & N \arrow[d] \arrow[r, phantom, "\cdots"] & N \arrow[d] \\
\Sigma^1 F_0 \arrow[from=2-1] \arrow[r] & \Sigma^2 F_1 \arrow[from=2-2] \arrow[r] & \Sigma^3 F_2 & \varinjlim \Sigma^{i+1} F_i \simeq *
\end{tikzcd}$$

zero on Htpy

Get $\varinjlim N_i \simeq N$

$$\begin{tikzcd}
N_0 \arrow[r] \arrow[d, equal] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & \cdots \\
W_0 & \Sigma^1 W_1 & \Sigma^2 W_2 &
\end{tikzcd}$$

$$\begin{tikzcd}
N_0 \wedge_E M \arrow[r] \arrow[d] & N_1 \wedge_E M \arrow[r] \arrow[d] & \cdots \\
W_0 \wedge_E M & \Sigma^1 W_1 \wedge_E M &
\end{tikzcd}$$

Get a SS

$$E^1_{s,t} = \pi_{t+s} \Sigma^s W_s \wedge_E M \Rightarrow \pi_{s+t} N \wedge_E M$$

$$\pi_t W_s \wedge_E M \cong \pi_* W_s \otimes_{E_*} M$$

$$E^2_{s,t} = \mathrm{Tor}^{E_*}_{s,t}(N_*, M_*)$$

Case (b) Want

$/ \; F_0 \; \pi(t) \; / \; F_2 \; \pi(t) \; / \; F_3$

Case (b)

Want

$$\begin{tikzcd}
 & & F_0 \arrow[dl] & & F_2 \arrow[dl] & & F_3 \arrow[dl] & \\
X & W_0 \arrow[l, "(\dagger)"'] & & W_1 \arrow[ll] \arrow[ul, "(\dagger)"'] & & W_2 \arrow[ll] \arrow[ul, "(\dagger)"'] & & \cdots \arrow[ll]
\end{tikzcd}$$

s.t. (1) E_*W_i is E_*-projective

(2) $E_*(\dagger)$ are surjective

Lemma: Given X, get $W \to X$ s.t. $E_*W \to E_*X$ epi, E_*W proj..

$$\bigvee_i S^{n_i} \longrightarrow E \wedge X$$

$$\begin{tikzcd}
S^{n_i} \arrow[rr] \arrow[dr] & & E \wedge X \\
 & E_\alpha \wedge X \arrow[ur] &
\end{tikzcd}$$

$$\Updownarrow$$

$$S^{n_i} \wedge DE_\alpha \longrightarrow X$$