WARPED PRODUCT METRICS ON $\mathbb{R}^2$

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Let M be $\mathbb{R}^2$ equipped with the metric :

$$||\frac{\partial}{\partial x}|| = 1$$

$$||\frac{\partial}{\partial y}|| = f(x)$$

and

$$< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >= 0$$

1. The Levi-Civita Connection

We did this calculation in class. To quickly recap, by metric compatibility, we know:

$$0 = \frac{\partial}{\partial y} f(x)^2 = \frac{\partial}{\partial y} < \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= 2 < \frac{\partial}{\partial y}, D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} >$$

$$0 = \frac{\partial}{\partial x} 1 = \frac{\partial}{\partial x} < \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >= 2 < \frac{\partial}{\partial x}, D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} >$$

$$2f(x)f'(x) = \frac{\partial}{\partial x} f(x)^2 = \frac{\partial}{\partial x} < \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= 2 < \frac{\partial}{\partial y}, D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} >$$

$$0 = \frac{\partial}{\partial y} 1 = \frac{\partial}{\partial y} < \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >= 2 < \frac{\partial}{\partial y}, D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} >$$

$$0 = \frac{\partial}{\partial y} 0 = \frac{\partial}{\partial y} < \frac{\partial}{\partial y}, \frac{\partial}{\partial x} >=< D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}, \frac{\partial}{\partial x} > + < \frac{\partial}{\partial y}, D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} >$$

$$0 = \frac{\partial}{\partial x} 0 = \frac{\partial}{\partial x} < \frac{\partial}{\partial y}, \frac{\partial}{\partial x} >=< D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \frac{\partial}{\partial x} > + < \frac{\partial}{\partial y}, D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} >$$

from this, and using $D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}$, we solve to get:

$$< D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= 0, < D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}, \frac{\partial}{\partial x} >= -f(x)f'(x)$$

so $D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -f(x)f'(x) \frac{\partial}{\partial x}$

Likewise:

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{f'(x)}{f(x)}\frac{\partial}{\partial y}$$

and

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0$$

This determines the connection via linearity and the Leibnitz rule.

2. Geodesics

We will try to understand a geodesic $c(t)$, starting at a point (x_0, y_0) with initial tangent vector (u_0, v_0).

If $c(t) = (x(t), y(t))$ then $c'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$. The geodesic equation then becomes:

$$0 = D_{c'(t)}(x'(t)\frac{\partial}{\partial x}+y'(t)\frac{\partial}{\partial y}) = x''(t)\frac{\partial}{\partial x}+x'(t)D_{c'(t)}\frac{\partial}{\partial x}+y''(t)\frac{\partial}{\partial y}+y'(t)D_{c'(t)}\frac{\partial}{\partial y}$$

$$= x''(t)\frac{\partial}{\partial x}+y''(t)\frac{\partial}{\partial y}+x'(t)^2 D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}+y'(t)^2 D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}+2x'(t)y'(t)D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}$$

so

$$0 = (x''(t) - y'(t)^2 f(x(t))f'(x(t)))\frac{\partial}{\partial x} + (y''(t) + 2x'(t)y'(t)\frac{f'(x(t))}{f(x(t))}\frac{\partial}{\partial y}$$

which gives the system of equations:

$$x''(t) - y'(t)^2 f(x(t))f'(x(t)) = 0$$
$$y''(t) + 2x'(t)y'(t)\frac{f'(x(t))}{f(x(t))} = 0$$

If $y'(t) \neq 0$ we can divide the second equation by $y'(t)$ to get:

$$0 = \frac{y''(t)}{y'(t)} + 2x'(t)\frac{f'(x(t))}{f(x(t)} = \frac{\partial}{\partial t}\log|y'(t)| + \log f(x(t))^2$$

Which means that $|y'(t)|f(x(t))^2$ is constant in t, except possibly for intervals where $y'(t) = 0$. Evaluating at $t = 0$ gives that this constant value is $|v_0|f(x_0)^2$. If $v_0 \neq 0$ we can conclude that $y'(t) = v_0\frac{f(x_0)^2}{f(x(t)}^2$. Since this expression is never zero (given that $v_0 \neq 0$) we know that out geodesics either have $y'(t)$ always positive, always negative, or always zero. In the last case we can easily see that $c(t) = (x_0 + u_0 t, y_0)$ is a geodesic.

We can substitute this expression into the other equation to get that:

$$x''(t) - v_0^2 \frac{f(x_0)^4 f'(x(t))}{f(x(t)^3} = 0$$

after multiplying by $2x'(t)$, this becomes:

$$\frac{\partial}{\partial t}(x'(t)^2 + v_0^2 \frac{f(x_0)^4}{f(x(t))^2}) = 0$$

So $x'(t)^2 + v_0^2 \frac{f(x_0)^4}{f(x(t))^2}$ is also constant. As before, substituting $t = 0$ determines this constant to be $u_0^2 + v_0^2 f(x_0)^2$. Note that this is just the norm squared of (u_0, v_0), so that if we assume our geodesic to be parametrized at unit speed, we have $u_0^2 + f(x_0)^2 v_0^2 = 1$ and the equations of the geodesic reduce to:

$$y'(t) = v_0 \frac{f(x_0)^2}{f(x(t)^2}$$

and

$$x'(t)^2 + v_0^2 \frac{f(x_0)^4}{f(x(t))^2} = 1$$

If we solve the second equation, we get :

$$x'(t) = \pm \sqrt{1 - v_0^2 \frac{f(x_0)^4}{f(x(t))^2}}$$

Here the sign is determined by whether u_0 is positive or negative.

The sign of $x'(t)$ can only change when the expression inside the radical is zero, meaning that $f(x(t)) = |v_0| f(x_0)$. Thus we know that x monotonically increases or decreases until it reaches such a value. Assume that x is increasing and that x_1 is the smallest value larger than x_0 where $f(x) = |v_0| f(x_0)$. To understand the behavior of the geodesic near x_1, expand f in a power series around x_1, giving $f(x) = |v_0| f(x_0) + (x - x_1) f'(x_1) + \ldots$. Only the asymptotics of f around x_1 matter, so we only need the first non-zero term after the constant term,so lett $f(x) = |v_0| f(x_0) + C(x - x_1)^k + \ldots$ for some $k \geq 1$.

The equation above then becomes $x'(t) = \sqrt{\frac{2C}{f(x_1)}}(x_1 - x)^{\frac{k}{2}} + \ldots$, where the omitted terms go to zero as x goes to x_1 faster that $(x - x_1)^{\frac{k}{2}}$. Standard differential equation facts say that for x near x_1 the solutions to our equation will be close to this equation with higher terms dropped.

This truncated equation is easy to solve directly, although the form depends on k.

- $k = 1$: The solutions are $x = x_1 - (\sqrt{\frac{C}{2f(x_1)}}t + K)^2$ where K is an arbitrary constant. What this says is that the solution increases until it reaches a maximum of x_1 and then starts to decrease.
- $k = 2$: The solutions are $x = x_1 - Ke^{-\sqrt{\frac{C}{2f(x_1)}}t}$. These solution increase forever, approaching x_1 asymptotically (at an exponential rate).
- $k > 2$: The solutions are $x = x_1 - (\frac{1}{(\frac{k}{2}-1)\sqrt{\frac{C}{2f(x_1)}}t+K})^{\frac{k}{2}-1}$. These solution again increase forever, approaching x_1 asymptotically.

Thus we know that our geodesic either hits $x = x_1$ and turns around (if $f'(x_1) \neq 0$) or is asymptotic to the line $x = x_1$ when $f'(x_1) = 0$. The case of $x(t)$ decreasing is exactly the same. So, we have a complete description of the behavior of a geodesic:

Summary Suppose $c(t)$ is a geodesic starting at (x_0, y_0), parametrized at unit speed, with tangent vector (u_0, v_0) at $t = 0$. Let x_- and x_+ be the values below and above x_0 where $f(x) = |v_0|f(x_0)$. If $f'(x_-)$ and $f'(x_+)$ are non-zero then c oscillates between these vertical lines, with y monotonic (like a sideways sine curve). If one of $f'(x_-)$ or $f'(x_+)$ is zero, then c is asymptotic to that line as $t \to \pm\infty$ and is tangent to the other once in the middle (like a sideways bell curve). If both $f'(x_-)$ and $f'(x_+)$ are zero, then c is asymptotic to one as $t \to \infty$ and the other as $t \to -\infty$ (like the graph of tangent). Finally, if there is no x_- or x_+ then c goes to infinity in the x coordinate (there are various possibilities here depending on whether the other is a critical value, etc.)

Suppose f has a proper local maxima at x_0. There are some $a < x_0 < b$ nearby with $f(a) = f(b)$ and $f'(a)$ and $f'(b)$ both not zero. The "strip" between a and b contains an oscillating geodesic and the vertical geodesic $x = x_0$. Since these cross infinitely many times, there are many pairs of points connected by more than one geodesic segment. Thus, under these assumptions geodesics are not determined by their endpoints, and, in particular, the exponential map is not a diffeomorphism. Can you determine the cut loci look like? Does something similar hold for a local maxima? If f is monotonic are geodesics uniquely determined by their endpoints? Is the exponential map a diffeomorphism? Look at the discussion of negatively curved metrics for some relevant discussion.

3. Curvature

Using the results about the connection, and the expression

$$\kappa = \frac{< R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y}, \frac{\partial}{\partial x} >}{||\frac{\partial}{\partial x}||^2||\frac{\partial}{\partial y}||^2 - < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >^2}$$

we can get the sectional curvature. Recall that $R(U,V)W = D_U D_V W - D_V D_U W - D_{[U,V]}W$. Since we are using the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ which are come from coordinates, we have $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ so that term drops out, giving:

$$R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial y} = D_{\frac{\partial}{\partial x}} D_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} - D_{\frac{\partial}{\partial y}} D_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}$$

Substituting from our results on the connection, this is

$$D_{\frac{\partial}{\partial x}}(-f'(x)f(x)\frac{\partial}{\partial x}) - D_{\frac{\partial}{\partial y}}(\frac{f'(x)}{f(x)}\frac{\partial}{\partial y})$$

expanding using the Leibnitz rule gives:

$$(f''(x)f(x) + f'(x)^2)\frac{\partial}{\partial x} + f'(x)f(x)D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} - \frac{f'(x)}{f(x)}D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}$$

Again using the connection, this is:

$$-(f''(x)f(x) + f'(x)^2)\frac{\partial}{\partial x} + 0 + f'(x)^2\frac{\partial}{\partial x} = -f''(x)f(x)\frac{\partial}{\partial x}$$

So, using the expression from the beginning of the section:

$$\kappa = -\frac{f''}{f}$$

We can work out some facts here:

If M has constant curvature K, we must have $f'' = -Kf$.

If $K > 0$ this means f is a linear combination of $\sin\sqrt{K}x$ and $\cos\sqrt{K}x$. In particular, it is impossible for such an f to be positive on all of $\mathbb{R}^2$. So, we can only get constant positive curvature metrics on "strips" in $\mathbb{R}^2$, meaning pieces of the form $[a.b] \times \mathbb{R}$. The maximal width of such a strip depends on linearly on $K^{-\frac{1}{2}}$. (These "strips" are the universal cover of the two sphere with two antipodal points removed - see if you can prove this).

If $K = 0$ this means f is linear, so $f = Ax + B$. If $f > 0$ everywhere then $A = 0$ and f is constant. This means the metric is constant, so M is just $\mathbb{R}^2$ with a standard Euclidean distance. When $A \neq 0$ we get a flat metric on a half plane. (These are the universal covers of a infinite cone with the cone point removed, the value of A depending on the cone angle - again, you should be able to prove this).

Finally, if $K < 0$ we have f as a linear combination of $e^{\sqrt{-K}x}$ and $e^{\sqrt{K}x}$. This will be positive everwhere if both coefficients are non-negative and one is positive. (All of these metrics must be isometric to each other because they are all isometric to the hyperbolic plane by our classification result for metrics of constant curvature. Can you find an isometry? Can you give a description like the ones in the previous parts for the metrics on parts of the plane when some of the coefficients are negative?)

More generally, having positive curvature means $f'' < 0$. There are no functions which are smooth, everywhere positive, and have f'' always negative (this is a good calculus problem). In contrast, negative curvature means $f'' > 0$, and it is quite easy to find smooth functions with $f > 0$ and $f'' > 0$ everywhere. In these cases we know (from the Cartan-Hadamard theorem) that the exponential map is a diffeomorphism.

4. Some specific examples

The first example given in class was $f = \sqrt{1 + x^2}$. This function has only one critical point, at $x = 0$, and f is otherwise monotonic. Looking at our results about geodesics, we can conclude that geodesics are of three types: those that come in from infinity and "turn around" before before hitting $x = 0$ and go back out (sort of like sideways parabolas, although the shape is more exponential than quadratic), those that come in from infinity and are asymptotic to $x = 0$ (kind of like the graph of $y = 1/x$, although is more than one shape possible, including the line $x = 0$ itself), and those that extend infinitely in both directions (there's a variety of shapes here, including horizontal lines).

Can work out more precisely what the shapes are? Can you solve the geodesic equations exactly?

The curvature in this example is not constant, but is easily computed from the formula we obtained. It works out to be $-\frac{1}{1+x^2}$. Thus this is one example of a manifold of variable negative curvature. In this case the manifold "flattens out" as $x \to \pm\infty$.

Another natural example is $f(x) = e^x$. From the previous section we know that this has constant curvature -1, and so is another model for the hyperbolic plane. The geodesics here are even simpler since f' is never zero. The discussion in the section on geodesics shows that all geodesics are either horizontal lines, or they come in from $+\infty$

turn around and head back out. In that section we also saw that $y' f(x)^2$ is constant, which here means y' is a multiple of e^{-2x}, so as $x \to \infty$, y will have a limiting value - so the ends of these geodesics are asymptotic to horizontal lines. With a bit more work one can solve the equations completely in this case, which will give a relationship between the spacing of the horizontal asymptotes and the x value where the "turn" happens.