18.950 pset 8 sol'ns

Note Title 11/29/2009

11. The *Poincaré upper half-plane* is defined as the set $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ endowed with an abstractly given first fundamental form (or metric) $(g_{ij}) = \frac{1}{y^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Although this metric is not induced by a surface f in $\mathbb{R}^3$, one can nevertheless calculate the Christoffel symbols and the geodesics[14] as quantities of the intrinsic geometry, see Figure 4.9. Hint: The geodesics are the curves with constant x as well as the half-circles whose centers lie on the x-axis. Introduce appropriate polar coordinates.

$$g_{11} = g_{22} = \frac{1}{y^2} \qquad g^{11} = g^{22} = y^2$$

$$g_{12} = g_{21} = 0 \qquad g^{12} = g^{21} = 0$$

Notation $\quad \partial_x := \frac{\partial}{\partial u_i}$

$$\Gamma_{ij,k} = \frac{1}{2}\left(-\partial_k g_{ij} + \partial_i g_{jk} + \partial_j g_{ik}\right)$$

$$\Gamma_{11,1} = \frac{1}{2}\left(-\partial_1 g_{11} + \partial_1 g_{11} + \partial_1 g_{11}\right) = 0$$

$$\Gamma_{12,1} = \Gamma_{21,1} = \frac{1}{2}\left(-\partial_1 g_{12} + \partial_1 g_{12} + \partial_2 g_{11}\right) = -y^{-3}$$

$$\Gamma_{22,1} = \frac{1}{2}\left(-\partial_1 g_{22} + \partial_2 g_{12} + \partial_2 g_{12}\right) = 0$$

$$\Gamma_{11,2} = \frac{1}{2}\left(-\partial_2 g_{11} + \partial_1 g_{12} + \partial_1 g_{12}\right) = y^{-3}$$

$$\Gamma_{12,2} = \Gamma_{21,2} = \frac{1}{2}\left(-\partial_2 g_{12} + \partial_1 g_{22} + \partial_2 g_{12}\right) = 0$$

$$\Gamma_{22,2} = \frac{1}{2}\left(-\partial_2 g_{22} + \partial_2 g_{22} + \partial_2 g_{22}\right) = -y^{-3}$$

$\Gamma^1_{11} = 0$ $\qquad$ $\Gamma^2_{11} = y^{-1}$

$\Gamma^1_{12} = -y^{-1}$ $\qquad$ $\Gamma^2_{12} = 0$

$\Gamma^1_{22} = 0$ $\qquad$ $\Gamma^2_{22} = -y^{-1}$

$$\gamma(t) = (x(t), y(t))$$

γ is a geodesic $\Longleftrightarrow$ $\nabla_{\dot\gamma}\dot\gamma = 0$

$$\dot\gamma = \dot x\,\partial_1 + \dot y\,\partial_2$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_k \left(\ddot u^k(t) + \sum_{i,j} \dot u^i(t)\dot u^j(t)\Gamma^k_{ij}(c(t))\right)\frac{\partial f}{\partial u^k},$$

$$= \left(\ddot x - \frac{2}{y}\dot x\dot y\right)\partial_1 + \left(\ddot y + \frac{1}{y}(\dot x^2 - \dot y^2)\right)\partial_2$$

So $\quad \ddot x - \frac{2}{y}\dot x \dot y = 0 \qquad (*)$

$\quad \ddot y + \frac{1}{y}(\dot x^2 - \dot y^2) = 0 \qquad (**)$

<u>CASE I</u>: γ is a vertical line $\Rightarrow$ $\dot x = 0$

<u>Get</u>: $(**) \Rightarrow \ddot y - \frac{1}{y}\dot y^2 = 0$

This ordinary diff'l eqn has a unique soln $y(t)$ given initial conditns.

$\Longrightarrow$ $\gamma(t) = (0, y(t))$ is a geodesic (vertical line)

CASE II: γ is a semicircle w/ endpoints on x-axis

WLOG: center of semicircle $= (0,0)$

(Since metric is clearly invariant under x-translation)

Write

$$x(t) = r\cos\theta(t) \qquad \dot{x}(t) = (-r\sin\theta)\dot{\theta}$$

$$y(t) = r\sin\theta(t) \qquad \dot{y}(t) = (r\cos\theta)\dot{\theta}$$

where r is constant

$$\ddot{x}(t) = (-r\cos\theta)\dot{\theta}^2 - (r\sin\theta)\ddot{\theta}$$

$$\ddot{y}(t) = (-r\sin\theta)\dot{\theta}^2 + (r\cos\theta)\ddot{\theta}$$

get

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} =$$

$$(-r\cos\theta)\dot{\theta}^2 - (r\sin\theta)\ddot{\theta} + 2(r\cos\theta)\dot{\theta}^2$$

$$= -(r\sin\theta)\ddot{\theta} + (r\cos\theta)\dot{\theta}^2$$

$$= -r\sin\theta\left(\ddot{\theta} - \frac{\cos\theta}{\sin\theta}\dot{\theta}^2\right) = 0$$

$$\ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2)$$

$$= (-r\sin\theta)\dot{\theta}^2 + (r\cos\theta)\ddot{\theta} + \frac{r^2\dot{\theta}^2}{r\sin\theta}(\sin^2\theta - \cos^2\theta)$$

$$= (r\cos\theta)\ddot{\theta} - r\frac{\cos^2\theta}{\sin\theta}\dot{\theta}^2$$

$$= (r\cos\theta)\left(\ddot{\theta} - \frac{\cos\theta}{\sin\theta}\dot{\theta}^2\right) = 0$$

Let $\theta(t)$ be a solution to the ordinary diff'l eqn:

$$\ddot{\theta} - \frac{\cos\theta}{\sin\theta}\,\dot{\theta}^2 = 0$$

(θ exists, and is uniquely defined by initial conditions)

$$\Rightarrow \begin{cases} x(t) = r\cos(\theta(t)) \\ y(t) = r\sin(\theta(t)) \end{cases}$$ satisfy (*), (**)

$\Rightarrow$ $\gamma(t)$ is a geodesic.

<u>Claim</u>: These are ALL of the geodsics

<u>Now</u>: given a point $(x,y) \in \mathbb{R}^2$, $y>0$ and a tangent vector $X \in T_{(x,y)}\mathbb{R}^2$

$$X = X^1\partial_1 + X^2\partial_2$$

Case I $\quad X' = 0$

Then $\exists$ vertical line geodesic

w/ $\dot{\gamma}(t_0) = X$

Case II $\quad X' \neq 0$

Then $\exists$ semi-circle geodesic

w/ $\dot{\gamma}(t_0) = X$

By uniqueness of geodesics, we have found all of them!

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12. Calculate the Gaussian curvature of the Poincaré upper half plane along the lines of 4.26 (ii).

4.26 (ii) states:

$$K = -\frac{1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right)$$

where in our case

$u = x$

$v = y$

$\lambda = y^{-2}$

So

$$K = -\frac{y^2}{2}\left(\left(\frac{-2y^{-3}}{y^{-2}}\right)_y\right)$$

$$= y^2\left(y^{-1}\right)_y$$

$$= -\frac{y^2}{y^2} = \boxed{-1}$$

i.e. The hyperbolic plane has constant negative curvature!

13. Show that for $z = x + iy \in \mathbb{C}$ all transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in I\!R,\ ad-bc>0,$$

are *isometries* of the Poincaré upper half-plane, i.e., preserve the abstract first fundamental form g_{ij} above.

The first fundamental form is

$$I = ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$$

write $\quad z = x + iy$

$$\bar{z} = \bar{x} + i\bar{y}$$

$$z = \frac{a\bar{z} + b}{c\bar{z} + d} \qquad \underbrace{ad - bc}_{=\,\Delta} > 0$$

$$\frac{a\bar{z}+b}{c\bar{z}+d} = \frac{a(\bar{x}+i\bar{y}) + b}{c(\bar{x}+i\bar{y}) + d} \; \frac{c(\bar{x}-i\bar{y}) + d}{c(\bar{x}-i\bar{y}) + d}$$

$$= \frac{ac(\bar{x}^2+\bar{y}^2) + bd + (ad+bc)\bar{x} + i(ad-bc)\bar{y}}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\,\bar{x} + d^2}$$

So!

$$x = \frac{ac(\bar{x}^2+\bar{y}^2) + bd + (ad+bc)\bar{x}}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}$$

$$y = \frac{(ad-bc)\bar{y}}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}$$

We compute:

$$\frac{1}{y^2}\left(dx^2 + dy^2\right)$$

$$= \frac{1}{\Delta^2} \frac{\left(c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2\right)^2}{\bar{y}^2}$$

$$\cdot \left[\left(\frac{\partial x}{\partial \bar{x}} d\bar{x} + \frac{\partial x}{\partial \bar{y}} d\bar{y}\right)^2 + \left(\frac{\partial y}{\partial \bar{x}} d\bar{x} + \frac{\partial y}{\partial \bar{y}} d\bar{y}\right)^2\right]$$

$$= \frac{1}{\Delta^2} \cdot \frac{\left(c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2\right)^2}{\bar{y}^2}$$

$$+ 2ac^2d\bar{x}^2 - (ad+bc)c^2\bar{x}^2$$
$$+ 2acd^2\bar{x} - 2bc^2d\bar{x}$$
$$+ ad^3 - ac^2d\bar{y}^2 + bc^3\bar{y}^2$$
$$- bcd^2$$

$$\left[\left(\frac{\left(2ac\bar{x} + (ad+bc)\right)\left(c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2\right)}{\left(c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2\right)^2}\right.\right.$$

$$= \begin{array}{c} c^2\Delta\bar{x}^2 \\ 2cd\Delta\bar{x} - c^2\Delta\bar{y}^2 \\ + d^2\Delta \end{array}$$

$$\left. - \frac{\left(ac(\bar{x}^2+\bar{y}^2) + bd + (ad+bc)\bar{x}\right)\left(2c^2\bar{x} + 2cd\right)}{\left(c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2\right)^2}\right) d\bar{x}$$

$$+\left(\frac{2ac\bar{y}\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)}{\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)^2}\right.$$

$$2acd^2\bar{y}-2bc^2d\bar{y}+2ac^2d\,\bar{x}\bar{y}-2bc^3\bar{x}\bar{y}$$

$$=\boxed{2cd\Delta\bar{y}+2c^2\Delta\bar{x}\bar{y}}$$

$$\left.-\frac{\left(ac\,(\bar{x}^2+\bar{y}^2)+bd+(ad+bc)\bar{x}\right)2c^2\bar{y}}{\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)^2}\right)^2 d\bar{y}$$

$$+\left(-\frac{(ad-bc)\bar{y}\,(2c^2\bar{x}+2cd)}{\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)^2}\,dx\right.$$

$$\boxed{-\Delta\,(2c^2\bar{x}\bar{y}+2cd\bar{y})}$$

$$+\left(\frac{(ad-bc)\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)}{\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)^2}\right.$$

$$\Delta\,(c^2\bar{x}^2+2cd\bar{x}+d^2-c^2\bar{y}^2)$$

$$\left.\left.-\frac{(ad-bc)\bar{y}\,(2c^2\bar{y})}{\left(c^2(\bar{x}^2+\bar{y}^2)+2cd\bar{x}+d^2\right)^2}\right)dy\right)^2$$

$$= \frac{1}{\bar{y}^2}\left(\left(\frac{c^2\bar{x}^2 + 2cd\bar{x} - c^2\bar{y}^2 + d^2}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}\, d\bar{x} + \frac{2cd\bar{y} + 2c^2\bar{x}\bar{y}}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}\, d\bar{y}\right)^2 + \left(\frac{-2c^2\bar{x}\bar{y} - 2cd\bar{y}}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}\, d\bar{x} + \frac{c^2\bar{x}^2 + 2cd\bar{x} - c^2\bar{y}^2 + d^2}{c^2(\bar{x}^2+\bar{y}^2) + 2cd\bar{x} + d^2}\, d\bar{y}\right)^2\right.$$

Negative

SAME

$$= \frac{1}{\bar{y}^2}\left(\frac{c^4\bar{x}^4 + 4c^2d^2\bar{x}^2 + c^4\bar{y}^4 + d^4 + 4c^3d\bar{x}^3 + 2c^4\bar{x}^2\bar{y}^2 + 2c^2d^2\bar{x}^2 + 4c^3d\bar{x}\bar{y}^2 + 4cd^3\bar{x} + 2c^2d^2\bar{y}^2}{c^4(\bar{x}^4 + 2\bar{x}^2\bar{y}^2 + \bar{y}^4) + 4c^2d^2\bar{x}^2 + d^4 + 4c^3d\bar{x}(\bar{x}^2+\bar{y}^2) + 2c^2d^2(\bar{x}^2+\bar{y}^2) + 4cd^3\bar{x}}\, d\bar{x}^2 + 0\, dx\,dy + \left(\quad\right) d\bar{y}^2\right)$$

Same

$$= \frac{1}{\bar{y}^2}\left(d\bar{x}^2 + d\bar{y}^2\right)$$

(Man, that was worse than I imagined! Sorry about that...)

14. Let $\lambda(x)$ be a positive differentiable function. For an abstract surface of rotation with metric $ds^2 = dx^2 + \lambda^2(x)dy^2$ ("warped product metric"), calculate the Christoffel symbols and show that the x-lines are geodesics parametrized by arc length. What do the rest of the geodesics look like?

$$g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{22} = \lambda^2(x)$$

$$\Gamma_{11,1} = \frac{1}{2}\left(-\partial_1 g_{11} + \partial_1 g_{11} + \partial_1 g_{11}\right) = 0$$

$$\Gamma_{12,1} = \Gamma_{21,1} = \frac{1}{2}\left(-\partial_1 g_{12} + \partial_1 g_{12} + \partial_2 g_{11}\right) = 0$$

$$\Gamma_{22,1} = \frac{1}{2}\left(-\partial_1 g_{22} + \partial_2 g_{12} + \partial_2 g_{12}\right) = -\lambda\lambda'$$

$$\Gamma_{11,2} = \frac{1}{2}\left(-\partial_2 g_{11} + \partial_1 g_{12} + \partial_1 g_{12}\right) = 0$$

$$\Gamma_{12,2} = \Gamma_{21,2} = \frac{1}{2}\left(-\partial_2 g_{12} + \partial_1 g_{22} + \partial_2 g_{12}\right) = \lambda\lambda'$$

$$\Gamma_{22,2} = \frac{1}{2}\left(-\partial_2 g_{22} + \partial_2 g_{22} + \partial_2 g_{22}\right) = 0$$

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = 0$$

$$\Gamma^1_{12} = 0 \qquad \Gamma^2_{12} = \frac{\lambda'}{\lambda}$$

$$\Gamma^1_{22} = -\lambda\lambda' \qquad \Gamma^2_{22} = 0$$

$$\gamma(t) = (x(t), y(t))$$

γ is a geodesic $\Longleftrightarrow$ $\nabla_{\dot\gamma}\dot\gamma = 0$

$$\dot\gamma = \dot x \partial_1 + \dot y \partial_2$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_k \left(\ddot u^k(t) + \sum_{i,j} \dot u^i(t)\dot u^j(t)\Gamma^k_{ij}(c(t))\right)\frac{\partial f}{\partial u^k},$$

$$= \left(\ddot x - \lambda\lambda' \dot y^2\right)\partial_1 + \left(\ddot y + \frac{\lambda'}{\lambda}\dot x \dot y\right)\partial_2$$

So

$$\ddot x - \lambda\lambda' \dot y^2 = 0$$

$$\ddot y + \frac{\lambda'}{\lambda}\dot x \dot y = 0$$

γ is an x-line parametrized by arclength $\Longleftrightarrow$ $y(t) = \text{constant}$, $\ddot x = 0$

$\Longrightarrow$ above eq^{n}s are satisfied.

$\Rightarrow$ x-lines are geodesics.

The second part of this question is vague, and I should have clarified what the question was looking for.

Consequently, I'll accept almost anything for this.

[But, if you are curious, I've posted a note on the web-page that gives a very complete picture

"Warped product metrics on $\mathbb{R}^2$"

by Kevin Whyte]