9- Formal groups and BP: a survey

Note Title 10/13/2008

Let E be a homotopy commutative, homotopy assoc. Ring spectrum:

<u>TFAE:</u>

(1) R complex orientable $\left(\begin{array}{c} V \\ \downarrow \\ X \end{array} \text{ n-plane bundle} \Rightarrow u_E \in R^{2n}(X^V) \right)$

(2) There is a factorization:

$$
\begin{tikzcd}
S^2 \arrow[r, "\Sigma^2 u"] \arrow[d, phantom, "\wr"] & \Sigma^2 E & u: S \to E \\
\mathbb{C}P^1 \arrow[d] & & \\
\mathbb{C}P^\infty \arrow[uur, dashed, "x"'] & &
\end{tikzcd}
$$

(3) The AHSS

$$H^*(\mathbb{C}P^\infty; E_*) \Rightarrow E^*(\mathbb{C}P^\infty)$$

Collapses

to give $E^*(\mathbb{C}P^\infty) \cong E_*[[x]]$

$|x| = 2$

(4) $\exists$ map of (htpy) ring spectra

$$MU \xrightarrow[\Phi]{} E$$

$(1) \Longrightarrow (2)$

uses $\quad MU(1) = (\mathbb{C}P^\infty)^\xi \simeq \mathbb{C}P^\infty / \mathbb{C}P^0 \simeq \mathbb{C}P^\infty$

$$x = u_\xi = \text{thom class of } \xi$$

$(2) \Longleftrightarrow (3)$

$$E^2_{AHSS} = E_*[[x]]$$

$(2) \Longleftrightarrow x$ is a P.C.

Multiplicativity of AHSS $\Longrightarrow d_n(x^k) = 0.$

$(2) \Longrightarrow (4)$

$$\begin{tikzcd}
\Phi \in E^0(MU) \arrow[r, leftrightarrow] & \Phi_n \in E^{2n}(MU(n)) \arrow[d, "\text{“splitting principle”}"] \\
& x \wedge \cdots \wedge x \in E^{2n}(\underbrace{MU(1) \wedge \cdots \wedge MU(1)}_{n})
\end{tikzcd}$$

$(4) \Rightarrow (1)$

$$\Phi_n \in E^n(MU(n)) = u_{\xi_n}$$

Thom class for universal

Compatibility $\Longleftrightarrow \quad u_{\xi_n} \wedge u_{\xi_m} = u_{\xi_{n+m}}$

Complex orientation

choice of x $\Longleftrightarrow$ choice of Φ

$\Longleftrightarrow$ choice of u_E

$\Downarrow$

Formal group law

$$F_\Phi(x,y) \in E_*[[x,y]]$$

Def a (commutative, 1-dim'l) formal gp law $/R$ (ring)

is a power series

$$F(x,y) \in R[[x,y]]$$

s.t.

(1) $F(0,x) = F(x,0) = 0$

(2) $F(x, F(y,z)) = F(F(x,y),z)$

(3) $F(x,y) = F(y,x)$

__e.g.__

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow[\mu]{} \mathbb{C}P^\infty$$

classifies $\xi \boxtimes \xi \,/\, \mathbb{C}P^\infty \times \mathbb{C}P^\infty$

makes $\mathbb{C}P^\infty$ into htpy comm. H-space

get

$$\begin{array}{ccc} \mu^* : E^*(\mathbb{C}P^\infty) & \longrightarrow & E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ \cong & & \cong \\ E_*[[x]] & & E_*[[x,y]] \\ \Psi & & \Psi \\ x & \longmapsto & F_\Phi(x,y) = F_E(x,y) \end{array}$$

__e.g.__ $E = H\mathbb{Z}$

$$x = c_1(\xi) \in H^2(\mathbb{C}P^\infty)$$

$$F_{H\mathbb{Z}}(x,y) = x + y \qquad \text{"additive fml gp"}$$

__Lem__ If $\pi_* E$ is concentrated in even degrees

$\Longrightarrow$ E is complex orientable

(pf) AHSS for $\mathbb{C}P^\infty$ collapses for dim'l reasons !!

e.g. $\quad \pi_* KU = \mathbb{Z}[\beta, \beta^{-1}] \qquad |\beta| = 2$

$$F_{KU}(x,y) = x + y + \beta x y$$

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e.g.

$$\begin{tikzcd}
S^2 \arrow[r] \arrow[d] & \Sigma^2 MU = \Sigma^2\left(\varinjlim \Sigma^{-2n} MU(n)\right) \\
MU(1) \arrow[ur, dashed, "x_{can}"'] &
\end{tikzcd}$$

Thm (Quillen) $\quad F_{MU}(x,y)$ is the universal formal group law

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L = Lazard ring

$$\begin{array}{c} \mathrm{Ring}(L, R) \cong \{FGL\text{'s}/R\} \\ \parallel \\ \mathrm{Spec}(L)(R) = \mathrm{Map}(\mathrm{Spec}(R), \mathrm{Spec}(L)) \end{array}$$

[So $\mathrm{Spec}(L) = \mathcal{M}_{FGL}$ moduli space of formal gp laws]

Thm (Lazard)

$$L \cong \mathbb{Z}[x_1, x_2, \ldots]$$

$$F = f^* F_{univ} \qquad F_{univ}$$

$$\mathrm{Spec}(R) \xrightarrow[f]{} \mathcal{M}_{FGL}$$

<u>Note:</u> $f : R \longrightarrow T$

$$F = FGL/_R \qquad F(x,y) = \sum a_{ij} x^i y^j$$

$$(f^* F)(x,y) = \sum f(a_{ij}) x^i y^j$$

$$f^* F = FGL/_T$$

Quillen's theorem is saying:

(1) $\pi_* MU \cong L \qquad |x_i| = 2i$

(2) under this isomorphism

$$F_{MU} \cong F_{univ}$$

An isomorphism of formal gp laws

$F_1, F_2 / R$ $\qquad f(x) = \sum_{i \geq 1} b_i x^{i+1} \quad b_0 \in R^\times$

is $f(x) \in R[[x]]^\times$ $\qquad f : F_1 \longrightarrow F_2$

s.t. $F_2(f(x), f(y)) = f(F_1(x,y))$

f is strict iff $f'(0) = 1$

$\iff b_0 = 1$

$\boxed{\text{Note: given } f, F_1, F_2 \text{ is determined}}$

Thm (Quillen-Landweber-Novikov)

$|b_i| = 2i$

$$MU_*MU \cong L[b_1, b_2, \ldots]$$

$$\alpha \in \mathrm{Ring}(MU_*MU, R) \cong \left\{ (F_1, F_2, f) \;\middle|\; \begin{array}{l} F_1, F_2 = \text{FGL's}/R \\ f : F_1 \to F_2 \\ \text{strict iso.} \end{array} \right\}$$

$$\mathrm{Spec}(MU_*MU)(R)$$

$f_\alpha(x) = \sum \alpha(b_i) x^{i+1}$ $\quad F_2$ defined.

F_1 is classified by $\alpha|_L$

Note MU_*MU free over π_*MU $\Rightarrow$ MU_*, MU_*MU is a Hopf algebroid

$\Rightarrow$ $\big(\mathrm{Spec}(MU_*)(R), \mathrm{Spec}(MU_*MU)(R)\big)$ is a groupoid.

Cor: $\big(\mathrm{Spec}(MU_*)(R), \mathrm{Spec}(MU_*MU)(R)\big)$

$=$ Groupoid $\left(\begin{array}{ll} \text{objects:} & \text{morphisms:} \\ \text{FGL's}/R, & \text{strict isos} \end{array} \right)$

From this, can determine

η_L, η_R, ψ etc....

Recursively (formulas cannot be written in closed form)

BP: "p-local version of MU"

$$F(x,y) =: x +_F y = x + y + \sum_{i+j\geq 2}' a_{ij} x^i y^j$$

assoc! $x +_F y +_F z$

$\Longrightarrow$ any power series $f(x) \in R[[x]]$

admits a unique expression as

$$f(x) = a_0 +_F a_1 x +_F a_2 x^2 +_F \cdots$$

$$= \sum^F a_i x^i$$

$n \in \mathbb{N}$

$[n]_F : F \longrightarrow F$ endomorphism of F

$$[n]_F(x) = \underbrace{x +_F x +_F \cdots +_F x}_{n} = nx + \cdots$$

Def: a FGL F/R is p-typical iff

$\exists\ v_i \in R$

s.t. $[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F v_3 x^{p^3} +_F \cdots$

Thm: Suppose R is a $\mathbb{Z}_{(p)}$-algebra.

Given $\{v_i\}$, $\exists!$ p-typical formal gp $F/_R$

s.t. $[p]_F(x) = \sum_i^F v_i x^{p^i}$

"Araki generators"

$\Rightarrow$ $V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$

carries a universal p-typical formal gp

$(F_{univ})_p$

$R = \mathbb{Z}_{(p)}$-alg

$\mathrm{Spec}(V)(R) = \{\text{p-typical FGL's}/_R\}$

Thm $R = \mathbb{Z}_{(p)}$-alg

"p-typicalization" F = formal gp law $/R$

$\exists$ functorial strict isomorphism

$$F \xrightarrow{\pi_p^F} F_p$$

when F_p is p-typical.

(s.t. If F is already p-typical

$$\pi_p^F = Id$$)

MU cx orientable

$$\Updownarrow$$

$$MU \xrightarrow[\mathrm{Id}]{} MU$$

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different orientations $\longleftrightarrow$ different choices of $x' \in MU^*(\mathbb{CP}^\infty)$

$$\cup\!\!\!\shortparallel$$

$$MU_{-*}[[x]]$$

Let $\pi_p^{univ} : F_{univ} \longrightarrow (F_{univ})_p$

$$x_p := \pi_p^{univ}(x) \in MU_{-*}[[x]]$$

$$\Downarrow$$

new orientation

idempotent ring map $\pi_p : MU \longrightarrow MU$

$$BP := \operatorname{colim}\left(MU \xrightarrow[\pi_p]{} MU \longrightarrow \cdots \right)$$

By construction, BP carries universal p-typical formal gp

$$\Longrightarrow BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \qquad |v_i| = 2(p^i - 1)$$

$\mathrm{Spec}(BP_*BP)(R) = \left\{ (F_1, F_2, f) \;\middle|\; \begin{array}{l} F_1, F_2 \quad p\text{-typical fgl's}/R \\ f : F_1 \to F_2 \quad \text{iso} \end{array} \right\}$

<u>lemma</u>:

$f \in R[[x]]$

$f : F_1 \longrightarrow F_2$

F_1 p-typical

F_2 is p-typical

$\Longleftrightarrow \quad f^{-1}(x) = \sum^{F_1} t_i x^{p^i}$

$\Longleftrightarrow \quad f(x) = \sum^{F_2} \bar{t}_i x^{p^i}$

So $\quad BP_*BP = BP_*[[t_1, t_2, \ldots]] \qquad (|t_i| = 2(p^i - 1))$

$\big(\mathrm{Spec}(BP_*)(R), \mathrm{Spec}(BP_*BP)(R)\big) = $ Hopf algebroid

objects: p-typical FGL's/R

morphisms: strict iso's

$$\eta_L : BP_* \longrightarrow BP_* BP$$
$$v_i \longmapsto v_i$$

$$\eta_R : BP_* \longrightarrow BP_* BP$$
$$v_i \longmapsto \eta_R(v_i)$$

Over BP_*BP $\exists$

$$\underset{\text{"}}{F_L} \qquad F_R$$

$$f_{univ} : F_{univ,p} \longrightarrow F^R_{univ,p}$$

$$f_{univ}^{-1}(x) = \sum_i{}^{F_{univ,p}} t_i x^{p^i}$$

<u>Lem</u>: Given a formal gp law F/R

$R = \mathbb{Q}$-alg

$\exists!$ strict iso

$\log_F : F \longrightarrow F_{add}$

$$F_{add}(x,y) = x+y$$

F p-typical $\iff$ $\log_F(x) = \sum m_i x^{p^i}$

$$\eta_L : BP_* \hookrightarrow BP_*[t_1, t_2, \dots]$$
$$v_i \longmapsto v_i$$

$$[p]_{F_L}(x) = \sum^{F_L} v_i x^{p^i}$$

Apply $\log_{F_L}$

$$p \sum m_i x^{p^i} = \sum_i \log(v_i x^{p^i})$$
$$= \sum_{i,j} m_j v_i^{p^j} x^{p^{i+j}}$$

Inductively get v_i in terms of $m_1, \dots, m_i$

$$m_i \in BP_* \otimes \mathbb{Q}$$

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e.g. $m_1 p^p + v_1 = p m_1$

$$\Rightarrow m_1 = \frac{v_1}{p - p^p}$$

$$[p]_{F_R}(x) = \sum^{F_R} \eta_R(v_i)\, x^{p^i}$$

$$\begin{tikzcd} F_L \arrow[dr, "\log_{F_L}"'] & F_R \arrow[l, "f^{-1}"'] \arrow[d, "\log_{F_R}"] \\ & F_{add} \end{tikzcd}$$

$$\log_{F_R}(x) = \sum \eta_R(m_i)\, x^{p^i}$$

$$\log_{F_L}(f^{-1}(x)) = \log_{F_R}(x) = \sum_i \eta_R(m_i)\, x^{p^i}$$

$$\shortparallel$$

$$\log_{F_L}\left(\sum^{F_L} t_i x^{p^i}\right)$$

$$\shortparallel$$

$$\sum m_j t_i^{p^j} x^{p^{i+j}}$$

So $$\eta_R(m_i) = \sum_{i_1+i_2=i} m_{i_1} t_{i_2}^{p^{i_1}}$$

e.g. $$\eta_R(m_1) = t_1 + m_1$$

$$\Longrightarrow \eta_R(v_1) = \eta_R\big((p-p^p)m_1\big) = (p-p^p)t_1 + v_1$$

$$\equiv v_1 + pt_1 \mod p^2$$

$\underline{\underline{\Psi}}$:

$$BP_*[t_1, t_2, \ldots) \xrightarrow[\psi]{} BP_*[t_1, \ldots, s_1, \ldots]$$

$$F_0 \xrightarrow[f_1]{} F_1 \xrightarrow[f_2]{} F_2$$

$$f_2^{-1} = \sum_i{}^{F_1} s_i x^{p^i}$$

$$f_1^{-1} = \sum_i{}^{F_0} t_i x^{p^i}$$

$$f_1^{-1}(f_2^{-1}(x)) = \sum_i{}^{F_0} \psi(t_i) x^{p^i}$$

$$\shortparallel$$

$$f_1^{-1}\left(\sum{}^{F_1} s_i x^{p^i}\right)$$

$$\shortparallel$$

$$\sum{}^{F_0} f_1^{-1}(s_i x^{p^i})$$

$$\shortparallel$$

$$\sum{}^{F_0} t_i s_j^{p^i} x^{p^{i+j}}$$

Apply $\log_{F_0}$

$$\sum m_i t_j^{p^i} s_k^{p^{i+j}} x^{p^{i+j+k}} = \sum m_j \psi(t_j)^{p^i} x^{p^{i+j}}$$

i.e.

$$m_1 + t_1 + s_1 = m_1 + \psi(t_1)$$

$$\Longrightarrow \psi(t_1) = t_1 \otimes 1 + 1 \otimes t_1$$

$$\psi(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1^p$$
$$+ \frac{v_1}{(p^{p-1}-1)} \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i}$$

Note $I = (p, v_1, v_2, \ldots\ldots)$

$$\psi(t_i) = \sum_{i_1+i_2=i} t_{i_1} \otimes t_{i_2}^{p^{i_1}}$$

So $\left(BP/I,\ BP_*BP/I\right) \cong P_*$ = poly part of dual Steenrod alg.

$$\eta_R(v_n) = v_n \mod (p, v_1, \ldots, v_{n-1})$$
$$\| $$
$$I_n$$

Inductively implies $\eta_R(I_n) \subset I_n BP_*BP$

Invariant

$\Longrightarrow$ BP_*/I_n is a (BP_*, BP_*BP) comodule

__Landweber__: $\{I_n\}$ are the only invariant prime ideals of BP_*