8. Low dim'l calculations: $\pi_* S$ at $p=2$

Note Title 10/6/2008

$\mathrm{Ext}_A^{s,t}$ $\quad t-s \leq 13$

in this range

$$E_1^{MSS} = \mathbb{F}_2[h_{1,0}, h_{1,1}, h_{1,2}, h_{1,3}, h_{2,0}, h_{2,1}, h_{2,2}, h_{3,0}]$$

$$d h_{ij} = \sum_{i_1+i_2=i} h_{i_1, j+i_2} h_{i_2, j}$$

Idea:

$$H^*(\mathbb{F}_2[h_{1,i}]) \Rightarrow H^*(\mathbb{F}_2[h_{1,i}, h_{2,i}]) \Rightarrow H^*(\mathbb{F}_2[h_{1,i}, h_{2,i}, h_{3,i}])$$

Using spectral sequences

$$H^*(\underbrace{\mathbb{F}_2[h_{1,i}, \ldots, h_{n-1,i}]}_{Y_{n-1}}) \otimes \mathbb{F}_2[h_{n,i}] \Rightarrow H^*(\overbrace{\mathbb{F}_2[h_{1,i}, \ldots, h_{n,i}]}^{Y_n})$$

"filter by powers of $h_{n,i}$"

$$H^*(Y_1) = \mathbb{F}_2[h_{1,i}] = \mathbb{F}_2[h_i]$$

Now: $d_2(h_{2,i}) = h_i h_{i+1}$

Get new relations: $h_i h_{i+1} = 0$

Get new cycles

$$b_{2,i} = h_{2,i}^2$$
$$e_{3,i} = \langle h_i, h_{i+1}, h_{i+2} \rangle$$

$$\Longrightarrow H^*(Y_2) = \mathbb{F}_2[h_i, b_{2,i}, e_{3,i}] \Big/ \text{rels}$$

rels

(1) $h_i e_{3,i+1} = e_{3,i} h_{i+3}$

(2) $(e_{3,i})^2 = h_i^2 b_{2,i+1} + b_{2,i} h_{i+2}^2$

(3) $e_{3,i} e_{3,i+1} = h_i h_{i+3} b_{2,i+1}$

(4) $h_{i+1} e_{3,i} = 0$

(5) $h_i h_{i+1} = 0$

$$\begin{array}{ccccccc} h_{1,i} & & h_{1,i+1} & & h_{1,i+2} & & h_{1,i+3} \\ & h_{2,i} & & h_{2,i+1} & & h_{2,i+2} & \\ & & h_{3,i} & & h_{3,i+1} & & \end{array}$$

$H^*(Y_3)$ $\quad e_{3,i} = d\, h_{3,i}$

<u>generators</u> come from previous generators, $h_{3,i}^2$, and rels in $H^*(Y_3)$ which read "$0=0$" once we set $e_{3,i}=0$

$h_i\ ,\ b_{2,i} \longrightarrow$ from before

$b_{3,i} = h_{3,i}^2$

$(1) \rightsquigarrow \quad e_{4,i} = \langle h_i, h_{i+1}, h_{i+2}, h_{i+3} \rangle$

$(4) \rightsquigarrow \quad h_i(1) = \langle h_{i+1}, h_i, h_{i+1}, h_{i+2} \rangle$

<u>Relations</u> (excluding those involving $e_{4,i}$)

$(2) \longrightarrow (1)' \qquad h_i^2\, b_{2,i+1} = b_{3,i}\, h_{i+2}^2$

$(3) \longrightarrow (2)' \qquad h_i h_{i+3}\, b_{2,i+1} = 0$

$(5) \longrightarrow (3)' \qquad h_i h_{i+1} = 0$

$(4)' \qquad h_i\, h_i(1) = h_{i+2}\, b_{2,i}$

$(5)' \qquad h_{i+2}\, h_i(1) = h_i\, b_{2,i+1}$

$(6)' \qquad h_i(1)^2 = h_{i+1}^2\, b_{3,i} + b_{2,i}\, b_{2,i+1}$

In our range:

$$h_0,\ h_1,\ h_2,\ h_3$$

$$b_{2,0},\ b_{2,1}$$

$$h_0(1),\ b_{30}$$

$$h_0^2\, b_{2,1} = b_{2,0}\, h_2^2$$

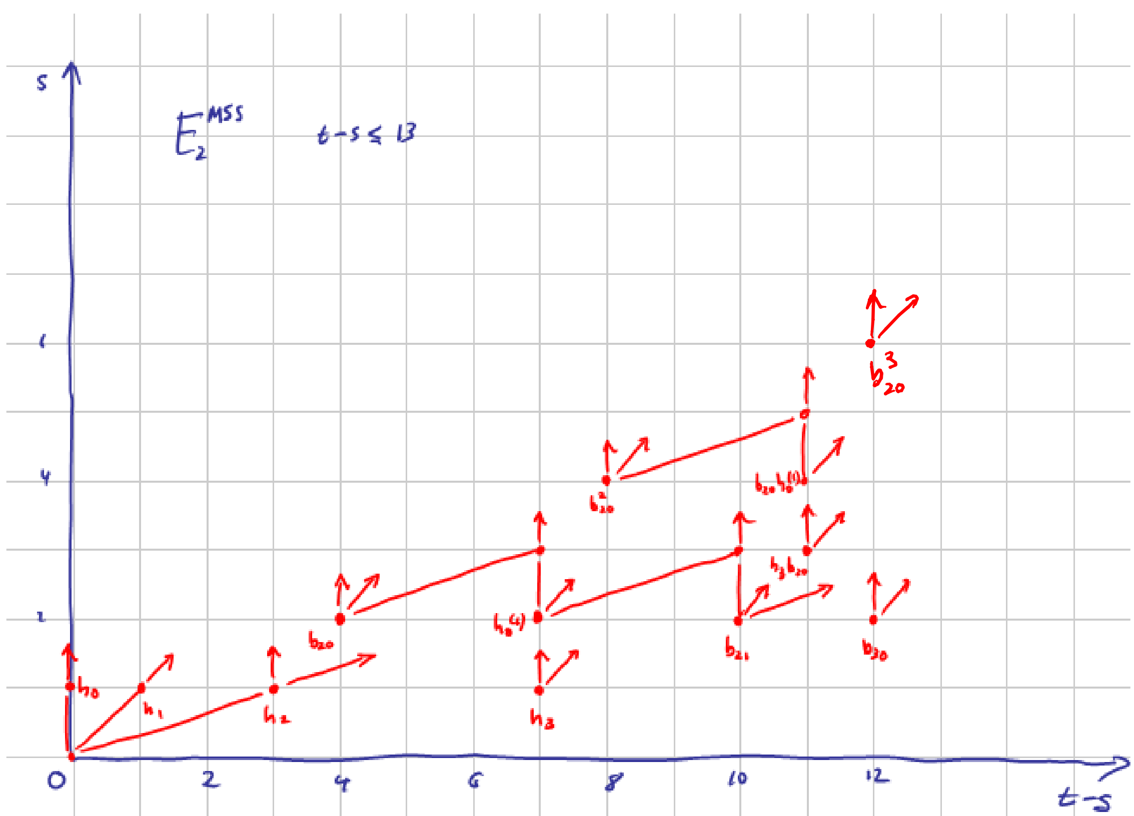
$$h_0 h_1 = 0$$

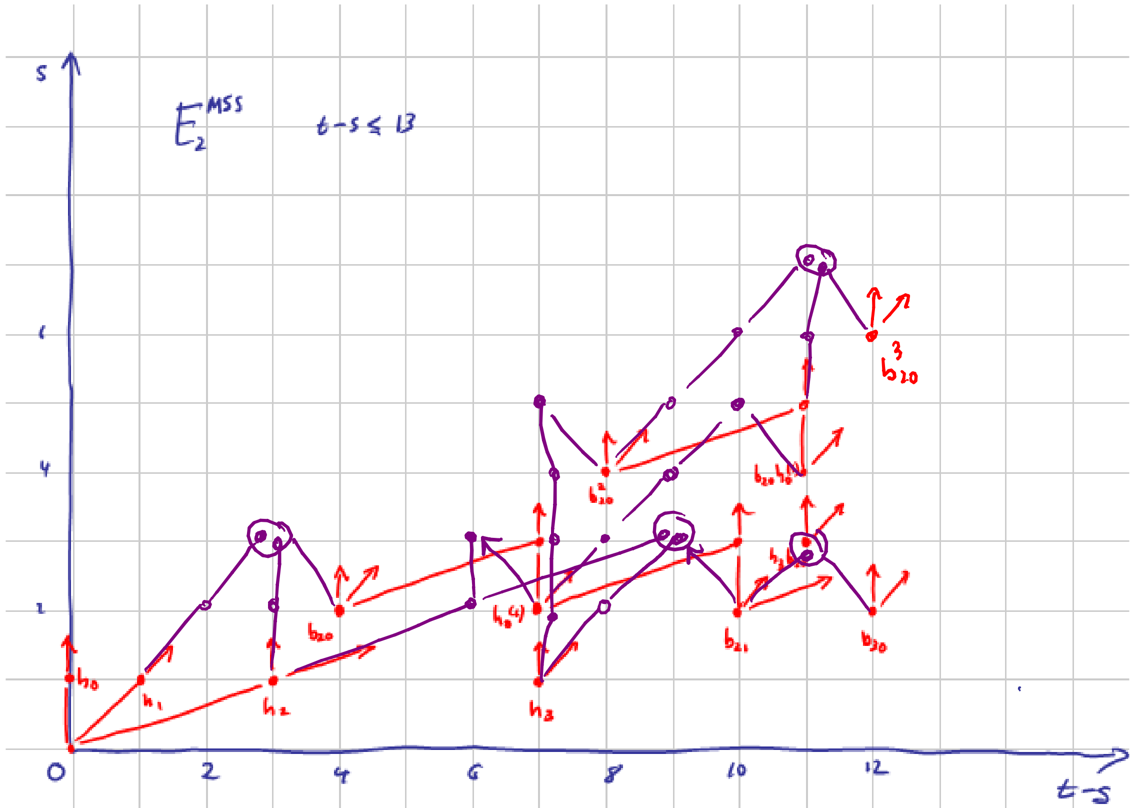
$$h_1 h_2 = 0$$

$$h_2 h_3 = 0$$

$$h_0\, h_0(1) = h_2\, b_{2,0}$$

$$h_2\, h_0(1) = h_0\, b_{2,1}$$





$$d_2(b_{20}) = d_2\left(Sq^1(h_{2,0})\right) = Sq^1(d_1 h_{20})$$

$$= Sq^1(h_{10} h_{11})$$

$$= h_{1,0}^2 h_{1,2} + h_{1,1} h_{1,1}^2$$

$$\boxed{d_2(b_{20}) = h_0^2 h_2 + h_1^3}$$

$$d_2\left(h_0 h_0(1)\right) = h_0 d_2 h_0(1)$$

$$\|$$

$$d_2(h_2 b_{20})$$

$$\|$$

$$h_2^2 h_0^2$$

$$\Longrightarrow \boxed{d_2 h_0(1) = h_0 h_2^2}$$

Same argument as b_{20}:

$$\boxed{d_2 b_{21} = h_1^2 h_3 + h_2^3}$$

$$d_2(b_{30}) = Sq^1(d_1 h_{30})$$

$$= Sq^1(h_{21} h_{10} + h_{12} h_{20})$$

$$= \underbrace{h_{21} h_1 + h_3 b_{20}} + \cdots$$

null in E_2

$$d_4(b_{20}^2) = Sq^2(d_2 b_{20})$$

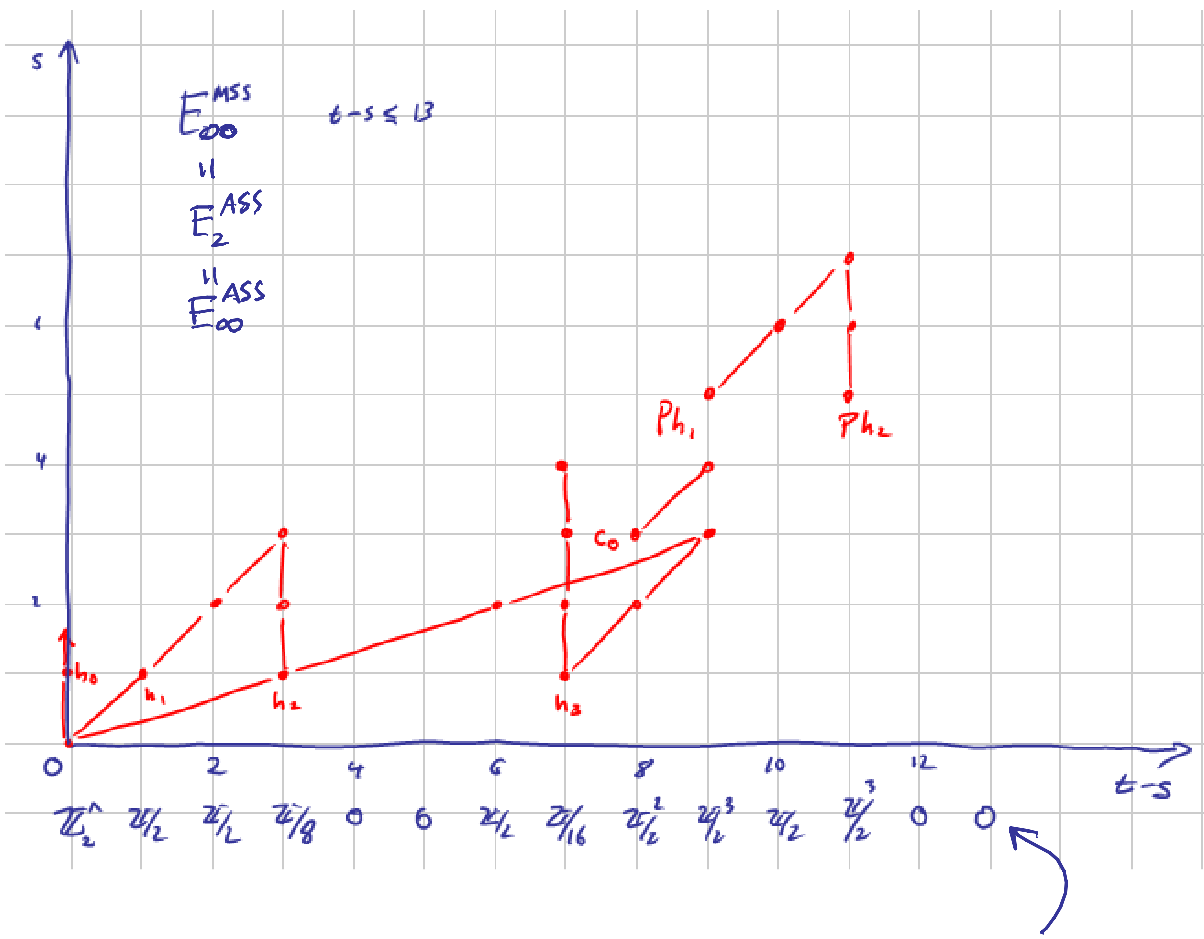
$$= Sq^2(h_0^2 h_2 + h_1^3)$$

$$= h_0^4 h_3$$

$$d_2 b_{20} h_0(1) = h_0(1)(h_0^2 h_2 + h_1^3)$$

$$+ \underbrace{b_{20} h_2^2 h_0}_{= h_0(1) h_2 h_0^2}$$

$$= h_0(1) h_1^3$$



$\pi_{t-s} S_2^\wedge$

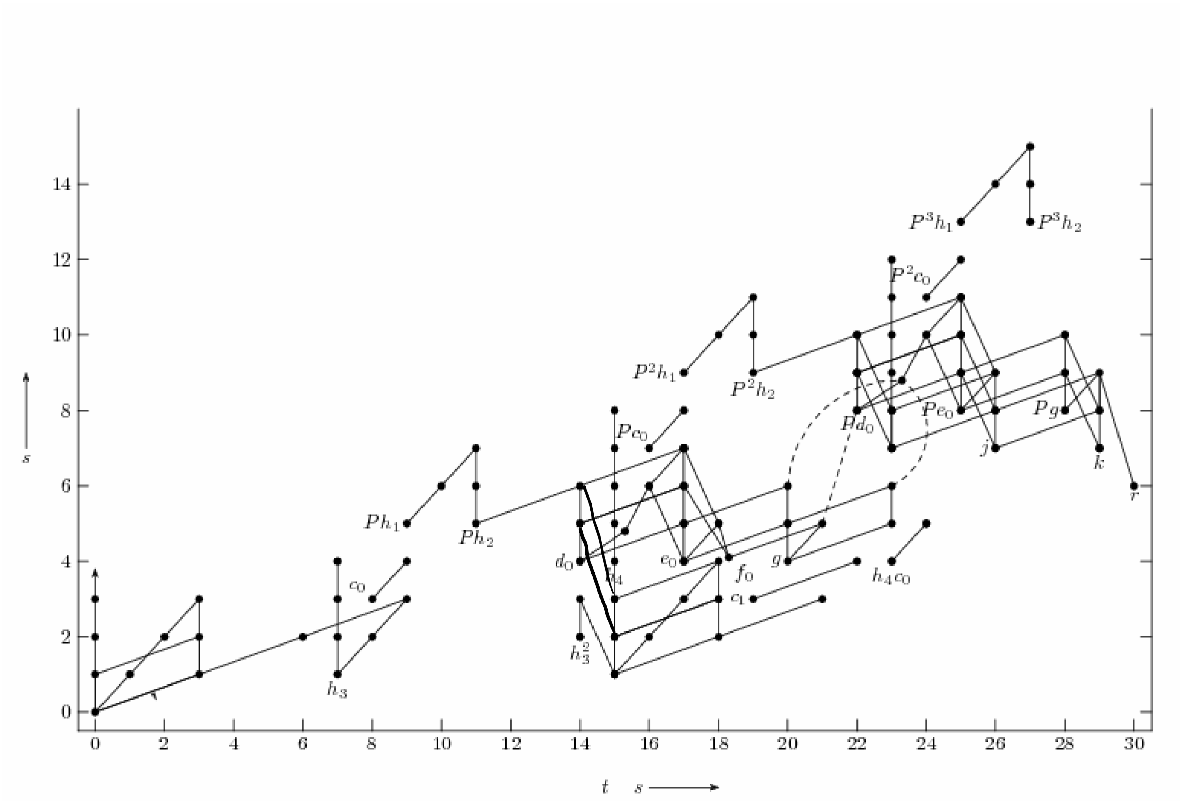
$c_0 = h_1 h_0(1)$

$Ph_1 = b_{20}^2 h_1$

$Ph_2 = b_{20}^2 h_2$

No room for Adams diff'ls

$\Longrightarrow \quad E_2^{ASS} = E_\infty^{ASS}$



Computing Adams diff'ls: Power operations & Bruner's formula

Power operations

$R =$ E_∞-ring spectrum

$\{\mathcal{E}(n)\} =$ E_∞ operad in unpointed spaces

$$\mathcal{E}(n) \underset{\Sigma_n}{\simeq} E\Sigma_n$$

$$\mathcal{E}(n)_+ \wedge R^{\wedge n} \xrightarrow[\mu_n]{} R \qquad (E_\infty\text{-structure maps})$$

$$S^k \xrightarrow[x]{} R$$

$$\begin{tikzcd} \Sigma^\infty E\Sigma_{2+} \wedge_{\Sigma_2} S^k \wedge S^k \arrow[r, "1\wedge x\wedge x"'] \arrow[rr, bend left, "P_2(x)"] & \mathcal{E}(2)_+ \wedge R^{\wedge 2} \arrow[r] & R \end{tikzcd}$$

$\rho : \Sigma_2 \to GL_2(\mathbb{R})$ reg rep

$$E\Sigma_{2+} \wedge_{\Sigma_2} S^{2k} \simeq \text{Thom}\left(\begin{array}{c} V_\rho = E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2 \\ \downarrow \\ B\Sigma_2 \end{array} \right)$$

$$\rho \simeq \text{sgn} \oplus 1$$

$\Longrightarrow V_\rho \simeq \xi \oplus 1$ $\quad \xi$ = canonical line bundle on $B\Sigma_2 = \mathbb{R}P^\infty$

So $\quad E\Sigma_{2+} \wedge_{\Sigma_2} S^k \wedge S^k \simeq (\mathbb{R}P^\infty)^{k\xi + k} \simeq \Sigma^k (\mathbb{R}P^\infty)^{k\xi}$

<u>Lemma:</u> $\quad (\mathbb{R}P^n)^{k\xi} \underset{\text{Homeomorphic}}{\approx} \mathbb{R}P^{n+k} / \mathbb{R}P^k$

(pf) <u>Idea:</u> Let $\mathbb{R}^{\text{sgn}}$ = sgn rep

$$\mathbb{R}P^n \simeq S(\mathbb{R}^{n(\text{sgn})}) / \Sigma_2$$

$$(\mathbb{R}P^n)^{k\xi} \approx \left(\frac{S(\mathbb{R}^{n(\text{sgn})}) \times D(\mathbb{R}^{k(\text{sgn})})}{S(\mathbb{R}^{n(\text{sgn})}) \times S(\mathbb{R}^{k(\text{sgn})})} \right) \Big/ \Sigma_2 \to \left(\frac{S(\mathbb{R}^{(n+k)\text{sgn}})}{S(\mathbb{R}^{k(\text{sgn})})} \right) \Big/ \Sigma_2 \simeq \frac{\mathbb{R}P^{n+k}}{\mathbb{R}P^k}$$

$\not{}$

Define: $$P_m^n = \Sigma^\infty \frac{\mathbb{RP}^n}{\mathbb{RP}^m} \simeq \left(\mathbb{RP}^{n-m}\right)^{m\xi}$$

has one cell in each dim $m \le i \le n$

this makes sense for negative n as well!

$$\begin{tikzcd}
\Sigma^k P_k^\infty \arrow[r, "P_k(x)"] & R \\
S^{2k} \arrow[u, hook, "\text{bottom cell}"] \arrow[ur, "x^2"'] &
\end{tikzcd}$$

"Higher cells carry power operations"

"Steenrod operations in ext $\Longrightarrow$ Power operations"

$x \in \pi_k(R)$ $\quad AF(x) = s$

$\vdots$

$2k+3$	0	$Sq^{s-3}(x)$
$2k+2$	0	$Sq^{s-2}(x)$
$2k+1$	0	$Sq^{s-1}(x)$
$2k$	0	$Sq^{s}(x)$

$\Sigma^k P_k^\infty$

Idea

$$\begin{array}{lcl} e_8 & \circ & Sq^0(h_3) = h_4 \\ & \cdot 2 & \\ e_7 & \circ & Sq^1(h_3) = h_3^2 \end{array} \qquad \Longrightarrow \qquad d\,h_4 = h_0 h_3^2$$

$\mathbb{R}P^\infty$

Bruner's Formula

Compression

$$x \in \mathrm{Ext}^{s,s+k}$$

$$d(Sq^i(x)) \text{ "=" } \sum_{j=i+1}^{s} \alpha^{k+s-i}_{k+s-j} \cdot Sq^j(x) + Sq^{i+r-1}(d_r x) + \alpha^{k+s-i}_{k-1} \cdot x\, d_r x$$

$k+s-i$ $\qquad$ $s+j+1$ $\qquad$ $s+2r+i-2$ $\qquad$ $2s+r+1$

Formula must be interpreted w/ care.

$d_r(Sq^i(x))$ is the "leading term"

α^m_n is the attaching map from the m-cell to the n-cell of $\mathbb{R}P^\infty$

Idea of Bruner's Formula

Adams resolution

$$\begin{tikzcd}
 & & & D^k \arrow[d, "x"] & \partial D^k \arrow[l] \arrow[d, "d_r x"] \\
S = F_0 \arrow[d] & F_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & F_s \arrow[l] & F_{s+r} \arrow[l] \\
K_0 & K_1 & & &
\end{tikzcd}$$

$$E\Sigma_2 = \bigcup E\Sigma_2^{[n]}$$

$$\partial\left(e_{s-i} \wedge D^k \wedge D^k\right) \simeq \partial e_{s-i} \wedge D^k \wedge D^k \cup e_{s-i} \wedge \partial D^k \wedge D^k$$

$$\cup \; e_{s-i} \wedge D^k \wedge \partial D^{k-1}$$

$$\begin{tikzcd}
e_{s-i} \wedge D^k \wedge D^k \arrow[d] \arrow[dddd, bend right, "Sq^i(x)"'] & \partial(e_{s-i} \wedge D^k \wedge D^k) \arrow[l] \arrow[dl, dashed, "\text{lift}"] & & \\
E\Sigma_2^{[s-i]} \wedge_{\Sigma_2} (D^k)^{\wedge 2} \arrow[dd] & \cdots \arrow[l, dashed] & E\Sigma_2^{[1]} \wedge_{\Sigma_2} (D^k)^2 \arrow[l] \arrow[dd] & E\Sigma_2^{[0]} \wedge_{\Sigma_2} D^k \wedge D^k \arrow[l] \arrow[d, equal] \\
 & & & D^k \wedge D^k \arrow[d, "x \wedge x = e_0 \wedge x \wedge x"] \\
F_{s+i} \arrow[d] & \cdots \arrow[l] & F_{2s-1} \arrow[l] & F_{2s} \arrow[l] \\
K_{s+i} & & &
\end{tikzcd}$$

lift will be detected on

One of $\quad \partial e_{s-i} \wedge S^k \wedge S^k \rightsquigarrow \alpha^{k+i-s}_{k+j-s} Sq^j(x)$

or $\quad e_{s-i} \wedge (\partial D^k \wedge D^k \cup D^k \wedge \partial D^k)$

$$\Big\downarrow \text{compression w/ } e_{s-i+1} \wedge \partial D_k \wedge D_k$$

$$e_{s-i+1} \wedge \partial D^k \wedge \partial D^k \longrightarrow Sq^i(dx)$$

$$\partial e_{s-i+1} \wedge \partial D_k \wedge D_k \longrightarrow \alpha^{k+i-s}_{k-1}\, x\, dx$$

if this attached to e_0

$$h_2^2 b_{30} \xleftarrow{Sq^1} h_{30} h_1^2$$

$$\shortparallel \qquad\qquad \shortparallel$$

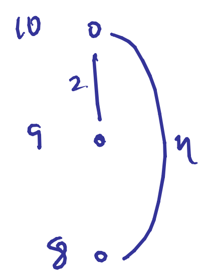
$$f_0 \qquad\qquad c_0$$

$h_1 \quad h_0 \quad h_1 \quad h_2$

$h_{20} \quad h_{20} \quad h_{21}$

$0 \quad h_{30}$

Note $\quad h_{30} h_1 \mapsto h_0(1)$



$Sq^1(c_0) = f_0$

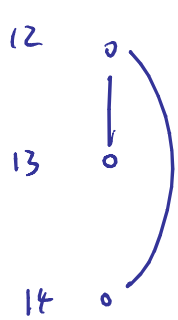
$Sq^2(c_0) = h_0 e_0$

$Sq^3(c_0) = c_0^2$

$\Rightarrow \quad d(f_0) = h_0^2 e_0$

$(Ph_1) \cdot e_0 = h_0^2 j$

$\Rightarrow$ bunch of d_2's



$Sq^2(d_0) = r$

$Sq^3(d_0) = 0 \quad \Rightarrow \quad \boxed{d_3(r) = h_1 d_0^2}$

$Sq^4(d_0) = d_0^2 = Pg$

Hopf inv 1 $\quad d_2(h_i) = h_0 h_{i-1}^2$

$\Rightarrow d_2(Sq^0 h_i) = h_0 h_i^2$

The diff'ls $d_r(h_0^i h_n)$ in general are resistant to Bruner's formula.

These are best seen using ANSS

BP, K-thy ----

Adams operations

$\updownarrow$

Vector fields on spheres

<u>J homomorphism</u>

$$SO(n) \longrightarrow \Omega^n S^n$$

$$(A: \mathbb{R}^n \to \mathbb{R}^n) \longmapsto (A^+: S^n \longrightarrow S^n)$$

gives $SO \longrightarrow \Omega^\infty \Sigma^\infty S^0 = \Omega^\infty S$

$$\pi_* SO \xrightarrow{J} \pi_* S$$

1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
	$0 \longmapsto \operatorname{Im} J$
3	$\mathbb{Z}$
	0
	0
	0
7	$\mathbb{Z}$

Show picture of image J

Bott periodicity $\rightsquigarrow$ v_1-periodicity

KO = ring spectrum

$S \longrightarrow KO$

$$\pi_* S \longrightarrow \pi_* KO$$

$$\begin{array}{l} \mathbb{Z} \\ \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \\ \mathbb{Z} \\ 0 \\ 0 \\ 0 \\ \mathbb{Z} \\ \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \\ \vdots \end{array}$$

Image of Hurewicz