7 - Nakamura's formula

Note Title 9/30/2008

Steenrod operations:

(Γ, A) = commutative Hopf algebroid

$\Longrightarrow$ $C^\bullet_\Gamma(A)$ = cosimplicial commutative alg

cobar complex

General theorem:

$R^\bullet$ = cosimp commutative alg

R^* = associated chain complex (DGA)

is an E_∞-algebra in cochain complexes.

[algebra over an E_∞-operad]

General thm

R^* = E_∞-alg in cochain complexes / k — fld char p

$H^*(R^*)$ has "Steenrod operations"

$Sq^0 \neq Id$ $\{Sq^i\}_{i \geq 0}$ and $\{P^i, \beta P^j\}_{i \geq 0, j \geq 0}$

$$Sq^i : H^s \longrightarrow H^{s+i}$$

$$P^i : H^s \longrightarrow H^{s+2i(p-1)}$$

$$\beta P^i : H^s \longrightarrow H^{s+2i(p-1)+1}$$

$\mathcal{E} = E_\infty$-operad $/ k$

<u>Construction</u> $\mathcal{E}(n)_* =$ chain complex of k-mod's

$\quad\;\; \mathcal{E}(n)^{-*}$ negatively graded cochain complex

[free over $k[\Sigma_n]$]

$$H^*(\mathcal{E}(n)^*) = \begin{cases} k, & n = 0 \\ 0, & o/w \end{cases}$$

<u>Rmk:</u>

$$H^s\left(\mathcal{E}(2)^* \otimes_{k[\Sigma_2]} k\right) = H_{-s}(C_2, k)$$

$$= \begin{cases} k\{e_s\}, & s < 0 \\ 0, & o/w \end{cases}$$

$$k[s] \xrightarrow{x} R^*$$

$$\begin{array}{ccc} \mathcal{E}(2)_* \otimes_{k[\Sigma_2]} k[2s] & \cong & \mathcal{E}(2) \otimes_{k[\Sigma_2]} k[s] \otimes k[s] \\ \downarrow & & \downarrow \\ R^* & \longleftarrow & \mathcal{E}(2) \otimes_{k[\Sigma_2]} R^* \otimes R^* \end{array}$$

get $H_i(\mathbb{Z}/2, k) \xrightarrow{P(x)} H^{2s-i}(R^*)$

$$e_i \longmapsto Sq^{s-i}(x)$$

Similarly: p odd $\quad |x| = 2s$

$$H_i(\Sigma_p, k) \xrightarrow{P(x)} H^{2ps-i}(R^*)$$

$$i = 2k(p-1) \longmapsto (-)\, P^{s-k}(x)$$

$$i = 2k(p-1)-1 \longmapsto (-)\, \beta P^{s-k}(x)$$

$|x| = 2s-1$

$|x| = 2s-1$

$$H_i(\Sigma_p, k[sgn]) \xrightarrow{P(x)} H^{(2s-1)p-i}(R^*)$$

$$i = (2k-1)(p-1) \longmapsto (-)\, P^{s-k}(x)$$

$$i = (2k-1)(p-1)-1 \longmapsto (-)\, \beta P^{s-k}(x)$$

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normalization "$(-)$" $= \quad (-1)^k \qquad |x| = 2s$

$$(-1)^k \left(\frac{p-1}{2}\right)! \qquad |x| = 2s-1$$

In general,

- Adem relations
- Cartan formula

$$Sq^{|x|}(x) = x^2$$

$$Sq^{>|x|}(x) = 0$$

|x| even:

$$P^s(x) = x^p$$

$|x| = 2s$

$$P^{>s}(x) = 0 \quad , \quad \beta P^{>s}(x) = 0$$

|x| odd:

$|x| = 2s-1$ (Hypotheses ?)

$$\beta P^{s-1}(x) = \langle \overbrace{x, \dots, x}^{p} \rangle$$

$$P^{\geq s}(x), \beta P^{\geq s}(x) = 0$$

Important fact: B = commutative Hopf alg $/k$

In $H^*(C^*_B(k))$

$p=2$

$$Sq^0([b_1|\dots|b_s]) = [b_1^2|\dots|b_s^2]$$

p odd

$$P^0([b_1|\dots|b_s]) = [b_1^p|\dots|b_s^p]$$

$\underline{\underline{p=2}}$

Thm (Nakamura) in May SS

$$Sq^i d_r x = d_{2r} Sq^i x$$

(pf) Idea

$$C^*_{A_\infty}(\mathbb{F}_2) = \left\{ \bigcup F_k \right\}_k$$

$$\shortparallel$$

$$C^*$$

$$\begin{tikzcd}
C^* & \cdots \arrow[l] & F_u \arrow[l] \arrow[d, phantom, "\ni" description] & F_{u-1} \arrow[l] & \cdots \arrow[l] & F_{u-r+1} \arrow[l] \arrow[d, phantom, "\bigcup" description] & \cdots \arrow[l] \\
 & & x \arrow[d, mapsto] & & & & \\
 & & dx \arrow[rrr, leftrightarrow] & & & y &
\end{tikzcd}$$

$$\Updownarrow$$

$$d_r(x) = y$$

$$\mathcal{E}(2) \otimes_{\mathbb{F}_2[\Sigma_2]} (C^*)^{\otimes 2} \longrightarrow C^*$$

$$\begin{array}{ccc} & & F_{2u-k} \\ & & \Psi \\ e_k \otimes x \otimes x & \longmapsto & Sq^{s-k}(x) \end{array}$$

$x \in C^s$

"Compression"

$$
\begin{tikzcd}
 & d\left(Sq^{s-k}(x)\right) & e_{k+1}\otimes dx\otimes x \arrow[d, mapsto] & \\
 & Sq^{s-k}(x) \arrow[u, mapsto] & e_k\otimes dx\otimes x + e_k\otimes x\otimes dx \arrow[ul, mapsto] & e_{k+1}\otimes dx\otimes dx \arrow[l, squiggly, mapsto] \\
 & \cap & \cap & \cap \\
\cdots & F_{2n-k} \arrow[l] & F_{2n-k-r+1} \arrow[l] & F_{2n-k-2r+2-1} \arrow[l]
\end{tikzcd}
$$

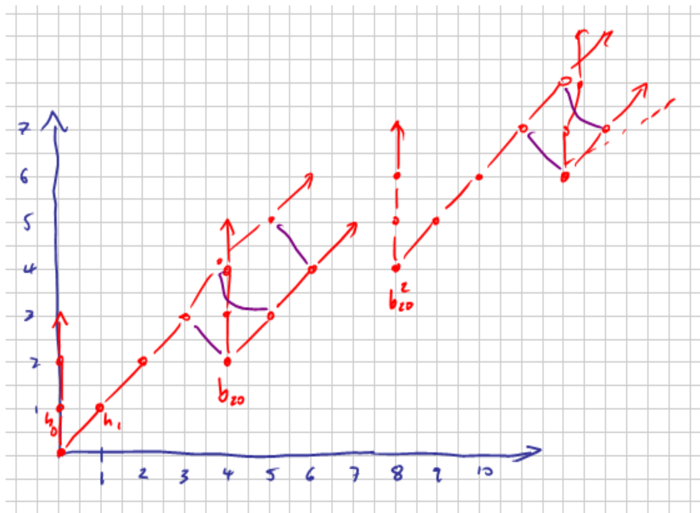
$$d(e_k\otimes x\otimes x) = \underset{-k}{e_k}\otimes \underset{n-r}{dx}\otimes \underset{n}{x} + \underset{-k}{e_k}\otimes \underset{n}{x}\otimes \underset{n-r}{dx}$$

$$d(e_{k+1}\otimes dx\otimes x) = e_{k+1}\otimes dx\otimes dx \;=\; Sq^{s-k}(dx)$$

$$+\, e_k\otimes dx\otimes x + e_k\otimes x\otimes dx$$

<u>So:</u> $$d_{2r}\left(Sq^{s-k}(x)\right) = Sq^{s-k}\left(d_r(x)\right)$$

e.g. MSS for $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$:



$$b_{20} = h_{20}^2 = Sq^1(h_{20})$$

$$d_1 h_{20} = h_{11} h_{10} = h_1 h_0$$

$$Sq^1(h_i) = h_i^2$$

$$Sq^0(h_i) = h_{i+1}$$

$$\boxed{\begin{array}{c} Sq^0(h_1) = [\xi_1^4] \\ \shortparallel \\ 0 \\ \text{in } A(1)_* \end{array}}$$

$$h_i = [\xi_1^{2^i}] \xrightarrow[Sq^0]{} [\xi_1^{2^{i+1}}] = h_{i+1}$$

$$d_2 b_{20} = d_2 Sq^1(h_{20}) = Sq^1(d_1 h_{20}) = Sq^1(h_1 h_0)$$
$$= Sq^1(h_1) Sq^0(h_0) + Sq^0(h_1) Sq^1(h_0)$$

$$= h_1 \cdot h_1^2 + 0$$

$$= h_1^3$$