Adams spectral sequences

Note Title 9/4/2008

Construction of ASS

X = spectrum, e.g. $X = S$

Input: $E_* X$

Output: "$\pi_* X$"

E = commutative Ring spectrum, $u : S \longrightarrow E$

[To simplify some technical details assume $E = A_\infty$-ring spectrum which is htpy commutative]

e.g. Spectrum = symmetric spectrum

$(Sp, \wedge)$ = symmetric monoidal category

E_∞-ring spectrum $\longleftrightarrow$ commutative monoid in $(Sp, \wedge)$

$E \wedge E \xrightarrow{\mu} E \qquad S \xrightarrow{\eta} E$

$$\begin{tikzcd} E\wedge E\wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & E\wedge E \arrow[d, "\mu"] \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "1\wedge\eta"] \arrow[dr, equal] & E\wedge E \arrow[d, "\mu"] & E \arrow[l, "\eta\wedge 1"'] \arrow[dl, equal] \\ & E & \end{tikzcd}$$

$$\left(\begin{tikzcd} E\wedge E \arrow[dr, "\mu"] \arrow[d, "\tau"'] & \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \quad \text{if } E_\infty \right)$$

[Warning: it is not known whether BP is an E_∞-ring spectrum]

but it is A_∞

$\rightsquigarrow$ cosimplicial spectrum $\quad E^{\wedge \bullet + 1}$

$$
\begin{tikzcd}
E \arrow[r, shift left=2, "u\wedge 1"] \arrow[r, shift right=2, "1\wedge u"'] & E\wedge E \arrow[l, "\mu"'] \arrow[r, shift left=3, "u\wedge 1\wedge 1"] \arrow[r, "1\wedge u\wedge 1"] \arrow[r, shift right=3, "1\wedge 1\wedge u"'] & E\wedge E\wedge E \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
S^\wedge_E := \operatorname*{holim}_{\Delta} E^{\wedge \bullet + 1} \\
S \arrow[u]
\end{tikzcd}
$$

More generally:

$$X^\wedge_E = \operatorname*{holim}_{\Delta} \left(E^{\wedge \bullet + 1} \wedge X\right)$$

$Y^\bullet$ = cosimplicial spectrum

$\pi_* Y^\bullet$ = cosimplicial abelian gp

$\rightsquigarrow$ cochain complex $\cdots \to \pi_* Y^n \xrightarrow{d} \pi_* Y^{n+1} \to \cdots$

$$d(x) = \sum_i (-1)^i d^i$$

<u>Bousfield-Kan Spectral Sequence</u>:

$$H^s(\pi_t Y^\bullet) \Longrightarrow \pi_{t-s} \operatorname{holim} Y^\bullet$$

Letting $Y^\bullet = E^{\wedge \bullet + 1} \wedge X$

$$E_2^{s,t} = H^s(\pi_t E^{\wedge \bullet + 1} \wedge X) \Longrightarrow \pi_{t-s} X^\wedge_E$$

Two questions:

(1) what is a homological description of $E_r^{s,t}$

(2) what is $X_E^\wedge$?

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(1) Assume

(*) E_*E is flat$/E_*$

Then (E_*E, E_*) is a "Hopf algebroid"

and $E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$

Hopf Algebroid

A commutative Hopf algebroid (Γ, A) has:

Γ, A = commutative rings

$$\left.\begin{array}{ll}
\text{Left unit} & \eta_L : A \longrightarrow \Gamma \\
\text{right unit} & \eta_R : A \longrightarrow \Gamma \\
\text{Coproduct} & \Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma \\
\text{conjugation} & c : \Gamma \longrightarrow \Gamma
\end{array}\right\} \text{maps of rings}$$

$$\text{augmentation} \quad \varepsilon : \Gamma \longrightarrow A$$

$$\Gamma = A \underset{\eta_L}{\circlearrowright} \Gamma \underset{\eta_R}{\circlearrowleft} A$$

Satisfying .... (Blah)

(Blah) is determined by the following proposition:

$\overset{X}{\overset{\shortparallel}{\mathrm{Spec}}}(A)$, $\overset{M}{\overset{\shortparallel}{\mathrm{Spec}}}(\Gamma)$ is an Internal groupoid in the category of schemes.

$$\text{Comm Rings}^{op} \xrightarrow[\simeq]{\text{Spec}} \text{Aff Schemes}$$

$$Z = \mathrm{Spec}(R) \in \text{Aff Schemes}$$

S = ring

$$\underset{\text{"S-points"}}{Z(S)} = \mathrm{Hom}(\mathrm{Spec}(S), \mathrm{Spec}(R)) \cong \mathrm{Hom}(R, S)$$

Yoneda lemma $\Longrightarrow$ map of affine schemes is determined by its effect on S-points

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We require

$(X(S), M(S))$ to be a groupoid $\mathcal{G}$

$$X(S) = \mathrm{Ob}\,\mathcal{G}$$

$$M(S) = \coprod_{a,b \in X(S)} \mathrm{Mor}_{\mathcal{G}}(a, b)$$

$$\begin{array}{ccc} \eta_L^* : M(S) & \longrightarrow & X(S) \\ \Psi & & \Psi \\ f: a \to b & \longmapsto & a \end{array}$$

$$\begin{array}{ccc} \eta_R^* : M(S) & \longrightarrow & X(S) \\ \Psi & & \Psi \\ f: a \to b & \longmapsto & b \end{array}$$

$$\begin{array}{ccc} \psi^* : M(S) \underset{X(S)}{\times} M(S) & \longmapsto & M(S) \\ \Psi & & \Psi \\ (f: a \to b,\ g: b \to c) & & g \circ f : a \to c \end{array}$$

$$\begin{array}{ccc} c^* : M(S) & \longrightarrow & M(S) \\ (f: a \to b) & \longmapsto & (f^{-1}: b \to a) \end{array}$$

$$\begin{array}{ccc} \varepsilon^* : X(S) & \longrightarrow & M(S) \\ \Psi & & \Psi \\ a & \longmapsto & Id : a \to a \end{array}$$

$$\Longrightarrow$$

deduce composition associativity

$\Rightarrow$ ψ coassociative

$$\begin{tikzcd} \Gamma \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A \Gamma \arrow[d, "1 \otimes \psi"] \\ \Gamma \otimes_A \Gamma \arrow[r, "\psi \otimes 1"'] & \Gamma \otimes_A \Gamma \otimes_A \Gamma \end{tikzcd}$$

etc....

Special case $A = k = $ fld, $\eta_L = \eta_R$

(Γ, k) = commutative Hopf algebra

Prop If E_*E is flat$/E_*$

then (E_*E, E_*) is a Hopf algebroid

Lem

$$\pi_* E \wedge E \wedge E \cong E_*E \otimes_{E_*} E_*E$$

(pf) $\quad E \wedge E \wedge E \simeq (E \wedge E) \wedge_E (E \wedge E)$

Künneth Spectral sequence (cf EKMM)

$$\mathrm{Tor}^{E_*}_{s,t}(E_*E, E_*E) \Rightarrow \pi_{s+t}\big((E \wedge E) \wedge_E (E \wedge E)\big)$$

$$\| $$

$$\begin{cases} E_*E \otimes_{E_*} E_*E, & s = 0 \\ 0, & s > 0 \end{cases}$$

$\square$

(pf of prop) The structure maps are given by:

$$\eta_L : E \cong E \wedge S \xrightarrow{1 \wedge \eta} E \wedge E$$

$$\eta_R : E \cong S \wedge E \xrightarrow{\eta \wedge 1} E \wedge E$$

$$\psi : E \wedge E \cong E \wedge S \wedge E \xrightarrow{1 \wedge \eta \wedge 1} E \wedge E \wedge E$$

$$c : E \wedge E \xrightarrow{\tau} E \wedge E$$

$$\varepsilon : E \wedge E \xrightarrow{\mu} E$$

//

Ext of Hopf algebroids.

$M \in \mathrm{Mod}_A$

M is a (left) (Γ, A)-comodule

$$\psi : M \longrightarrow \Gamma \otimes_A M$$ (map of left A-modules)

Counital

$$\begin{tikzcd} M \arrow[r] \arrow[dr, equal] & \Gamma \otimes_A M \arrow[d, "\varepsilon \otimes 1"] \\ & A \otimes_A M \end{tikzcd}$$

Coassociative:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A M \arrow[d, "1 \otimes \psi"] \\ \Gamma \otimes_A M \arrow[r, "\psi \otimes 1"'] & \Gamma \otimes_A \Gamma \otimes_A M \end{tikzcd}$$

Note: A is a Γ-comodule

$$A \xrightarrow[\eta_L]{} \Gamma \cong \Gamma \otimes_A A$$

X = Spectrum

$$\Longrightarrow E_* X \in \mathrm{CoMod}_{(E_*E,\, E_*)}$$

$$\psi : E \wedge X \cong E \wedge S \wedge X \xrightarrow[1 \wedge \eta \wedge 1]{} E \wedge E \wedge X \simeq (E \wedge E) \wedge_E (E \wedge X)$$

If Γ/A flat $\Longrightarrow$ $\mathrm{Comod}_{(\Gamma,A)}$ is abelian cat

<u>Adjoint pair</u>

$$\mathcal{U} : \mathrm{CoMod}_{(\Gamma,A)} \rightleftarrows \mathrm{Mod}_A : \Gamma \otimes_A -$$

Category $\mathrm{CoMod}_{(\Gamma,A)}$ has enough injectives

I = injective A-mod

$$\mathrm{Hom}_\Gamma(-, \Gamma \otimes_A I) \cong \mathrm{Hom}_\Gamma(-, I)$$

exact

$\Longrightarrow$ $\Gamma \otimes_A I$ is an injective

$$M \overset{\psi}{\hookrightarrow} \Gamma \otimes_A M \xhookrightarrow{\text{flatness}} \Gamma \otimes_A I$$

$\Longrightarrow$ there exists $\mathrm{Ext}^s_\Gamma(M,N)$.

- there are also notions of

graded commutative Hopf algebroids

$$ab = (-1)^{|a||b|} ba \qquad \text{e.g. } (E_*E, E_*)$$

- graded comodules

$$\Longrightarrow \quad \mathrm{Hom}_\Gamma(M,N) \subseteq \mathrm{Hom}_A(M,N) \quad \text{graded}$$

$$\Longrightarrow \quad \mathrm{Ext}^s_\Gamma(M,N) \quad \text{graded}$$

$$\mathrm{Ext}^s_\Gamma(M,N) \cong \bigoplus_t \mathrm{Ext}^{s,t}_\Gamma(M,N)$$

Computing Ext_Γ:

A Γ-comodule M is extended if

$$M \cong \Gamma \otimes_A N \quad \text{for some } N$$

$$\begin{tikzcd}
\mathrm{Mod}_A \arrow[rr, "\mathcal{U}", "\text{exact}"'] \arrow[dr, "\Gamma\otimes_A - \ \text{exact}"'] & & Ab \\
& \mathrm{CoMod}_{(\Gamma,A)} \arrow[ur, "\mathrm{Hom}_\Gamma(A,-)"'] &
\end{tikzcd}$$

"Grothendieck spectral sequence"

$$\Longrightarrow \quad \mathrm{Ext}^{s}_{\Gamma}(A, \Gamma \otimes_A N) = \begin{cases} N, & s = 0 \\ 0, & s > 0 \end{cases}$$

Standard homological algebra:

given a resolution by extended comodules

$$0 \to M \longrightarrow \Gamma \otimes_A N_0 \longrightarrow \Gamma \otimes_A N_1 \longrightarrow \cdots$$

Get spectral sequence

$$E_1^{s,t,k} = \mathrm{Ext}^{s,t}_{\Gamma}(A, \Gamma \otimes_A N_k) \Longrightarrow \mathrm{Ext}^{s+k,t}_{\Gamma}(A, M)$$

$$\cong$$

$$\begin{cases} N_k & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\Longrightarrow \quad \mathrm{Ext}^{s,t}_{\Gamma}(A, M) \cong H^{s,t}(N_*)$$

Cobar complex: $M \in \mathrm{CoMod}_\Gamma$

Form a split coaugmented cosimplicial Γ-comod:

$C^\bullet(\Gamma, \Gamma, M)$:

$$\begin{tikzcd}
M \arrow[r, "\psi"] & \Gamma \otimes_A M \arrow[l, bend left, dashed, "\varepsilon \otimes 1"] \arrow[r, shift left=2, "\psi \otimes 1"] \arrow[r, shift right=2, "1 \otimes \psi"'] & \Gamma \otimes_A \Gamma \otimes_A M \arrow[l, "1 \otimes \varepsilon \otimes 1"' description] \arrow[l, bend left, dashed, "\varepsilon \otimes 1 \otimes 1"] \arrow[r, shift left=3, "\psi \otimes 1 \otimes 1"] \arrow[r, "1 \otimes \psi \otimes 1" description] \arrow[r, shift right=3, "1 \otimes 1 \otimes \psi"'] & \Gamma \otimes_A \Gamma \otimes_A \Gamma \otimes_A M \arrow[l, shift right=1.5, "1 \otimes \varepsilon \otimes 1 \otimes 1"'] \arrow[l, shift left=1.5, "1 \otimes 1 \otimes \varepsilon"] \arrow[l, bend left, dashed, "\varepsilon \otimes 1 \otimes 1 \otimes 1"]
\end{tikzcd}$$

Exact: using "extra degeneracy"

cosimplicial abelian gp

$$C^\bullet(A, \Gamma, M) = \mathrm{Hom}_\Gamma(A, C^\bullet(\Gamma, \Gamma, M))$$

$\parallel$

$C^\bullet_\Gamma(M)$

$$\begin{tikzcd}
M \arrow[r, shift left=2, "\eta_R \otimes 1"] \arrow[r, shift right=2, "\psi"'] & \Gamma \otimes M \arrow[l, "\varepsilon \otimes 1" description] \arrow[r, shift left=3, "\eta_R \otimes 1 \otimes 1"] \arrow[r, "\psi \otimes 1" description] \arrow[r, shift right=3, "1 \otimes \psi"'] & \Gamma \otimes \Gamma \otimes M \cdots \arrow[l, shift left=1.5, "\varepsilon \otimes 1 \otimes 1"'] \arrow[l, shift right=1.5, "1 \otimes \varepsilon \otimes 1"]
\end{tikzcd}$$

We have:

Lemma

$$H^s(C^\bullet_\Gamma(M)) \cong \mathrm{Ext}^s_\Gamma(A, M)$$

Consider cosimplicial algebra sp

$$\pi_* E^{\wedge \bullet + 1} \wedge X$$

$$\pi_* E^{\wedge s+1} \cong \pi_* \underbrace{(E\wedge E)\wedge_E (E\wedge E)\wedge_E \cdots \wedge_E (E\wedge E)}_{s} \wedge_E E\wedge X$$

$$\cong E_*E^{\otimes_{E_*} s} \otimes_{E_*} E_*X$$

Lem

$$\pi_* E^{\wedge \bullet + 1} \cong C^\bullet_{E_*E}(E_*X)$$

Cor

$$E_2^{s,t} \cong H^{s,t}(\pi_* E^{\wedge \bullet + 1} \wedge X) \cong \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$$
