12 – Monochromatic layers

Note Title 11/4/2008

Algebraic Chromatic spectral sequence

$$\begin{array}{ccc}
\mathrm{Ext}^{s+n,t}_{BP_*BP}(BP_*) & \xRightarrow{\text{ANSS}} & \pi_{t-s-n} S_{(p)} \\
\Uparrow \text{ACSS} & & \Uparrow \text{CSS} \\
\mathrm{Ext}^{s,t}_{BP_*BP}(M_n(BP_*)) & \underset{\substack{\text{Monochromatic} \\ \text{ANSS}}}{\Longrightarrow} & \pi_{t-s-n} M_n S
\end{array}$$

explicitly:

$$\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
S_{E(2)} & \longleftarrow & M_2 S \\
\downarrow & & \\
S_{E(1)} & \longleftarrow & M_1 S \\
\downarrow & & \\
S_{\mathbb{Q}} & &
\end{array}$$

First we identify $M_n S$ . . . .

Nilpotence Thm (Devinatz - Hopkins - Smith)

Strongest form R = ring spectrum (homotopy unital)

$$\begin{array}{c} \pi_* R \xrightarrow{h} MU_* R \\ \rotatebox{90}{$\in$} \\ x \end{array}$$

x nilpotent $\iff$ $h(x)$ nilpotent

Consequence X = finite

$R = F(X,X)$ ring spectrum

$f : \Sigma^d X \longrightarrow X$ self map

f nilpotent $\iff$ $MU_* f$ is nilpotent

Nishida's thm

$\pi_{>0} S$ torsion

$\Rightarrow$ $\pi_{>0} S \xrightarrow[\text{zero}]{} MU_{>0} S$

$\Rightarrow$ $\pi_{>0} S$ consists of nilpotent elts

X finite, p-local

Suppose $f : \Sigma^d X \longrightarrow X$

non-nilpotent

$\Longleftrightarrow \quad BP_* f \neq 0$

Let n be minimal s.t.

$E(n)_* X \neq 0$ "X type n"

$\cong$

$BP_* X \otimes_{BP_*} E(n)$

$\boxed{\begin{array}{l} \text{Note: } E(n)_* X = 0 \ \forall n \\ \Rightarrow X_{E(n)} = 0 \\ \Rightarrow X \simeq * \end{array}}$

means $BP_* X$ is $v_0, v_1, \ldots, v_{n-1}$-torsion

has some v_n-non-torsion

$\Updownarrow$

$K(n)_* X \neq 0$

| Aside $X_{K(n)} \simeq X_{E(n)}$

But $K(n)_* f$ might still be nilpotent

Can argue using $BP_* f \neq 0$

that $K(m)_* f$ non-nilpotent for some m.

Periodicity thm: (Hopkins – Smith)

Suppose that X is type n

then $\exists$ $v : \Sigma^{d} X \longrightarrow X$

s.t. v is an $E(n)$-equivalence.

Rmk since $X_{E(n)} \simeq X_{K(n)}$

v is a $K(n)$-equivalence

$$v : K(n)_* X \xrightarrow{\cong} K(n)_* X$$

prop: $\exists$ i, j s.t.

$$K(n)_* v^i = \cdot\, v_n^j$$

Def: such a self map v is called a v_n-self map

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prop: v, v' two v_n-self maps

$\exists$ i, j s.t. $v^i \simeq (v')^j$.

Cor: $X[v_n^{-1}] := \mathrm{Tel}\left(X \xrightarrow[v]{} \Sigma^{-d} X \xrightarrow[v]{} \cdots\right)$

is unambiguous

Note

$$X \xrightarrow{\quad} X[v_n^{-1}]$$

$E(n)$-equivalence

Conj: Telescope conjecture:

$$X[v_n^{-1}] \longrightarrow X_{E(n)}$$

is an equivalence.

Important type n complexes

type n **means** BP_*

$v_0, \ldots, v_{n-1}$ torsion.

$$\begin{tikzcd}
S \arrow[r, "p^{i_0}"'] & S \arrow[d] \\
\Sigma^{2i_1(p-1)} M(p^{i_0}) \arrow[r, "v_1^{i_1}"'] & M(p^{i_0}) \arrow[d] \\
\Sigma^{2i_2(p^2-1)} M(p^{i_0}, v_1^{i_1}) \arrow[r, "v_2^{i_2}"'] & M(p^{i_0}, v_1^{i_1}) \arrow[d] \\
& M(p^{i_0}, v_1^{i_1}, v_2^{i_2})
\end{tikzcd}$$

etc ...

Note: $BP_* M(p^{i_0}, \ldots, v_n^{i_n}) \cong BP_* / (p^{i_0}, \ldots, v_n^{i_n})$

"Generalized Moore spectra"

$M(p, v_1, \ldots, v_n) = V(n)$ "Smith-Toda complex"

(when it exists ...)

"Smashing conj"

$$\forall X, \quad X_{E(n)} \simeq S_{E(n)} \wedge X$$

Note: this is NOT true for $K(n)$-localization.

e.g. $X_{K(1)} \simeq (X_{E(1)})^\wedge_p$

in general: $X^\wedge_p \not\simeq X \wedge S^\wedge_p$

$$\left[M^\wedge_p \not\cong M \otimes \mathbb{Z}^\wedge_p \right]$$

if M not f.g.

Computation of layers:

Goal: explicit understanding of $M_n S$

$$BP_* M_n(S) = ?$$

$$\begin{tikzcd}
BP \wedge M_n(S) \arrow[r] \arrow[d, "\wr"] & BP \wedge S_{E(n)} \arrow[r] \arrow[d, "\wr"] & BP \wedge S_{E(n-1)} \arrow[d, "\wr"] \\
M_n BP \arrow[r] & BP_{E(n)} \arrow[r] & BP_{E(n-1)}
\end{tikzcd}$$

$E(n-1) \vee K(n)$ equiv $\Rightarrow$ $E(n)$-equiv.

$K(n)$-equivalence

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[ddr, bend right] & & \\
& Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_{K(n)} \arrow[d] \\
& X_{E(n-1)} \arrow[r] & X_{K(n), E(n-1)}
\end{tikzcd}$$

$E(n-1)$-equiv.

$E(n-1)$-equiv

$S_{E(n-1)} \wedge (K(n)\text{-equiv})$

$\Longrightarrow$ $K(n)$-equiv

Htpy pullback of $E(n)$-local spectra

$\longrightarrow$ $E(n)$-local

Thm: Chromatic fracture X = spectrum

$$\begin{tikzcd} M_n X \arrow[r, "\simeq"] \arrow[d] & M_n(X_{K(n)}) \arrow[d] \\ X_{E(n)} \arrow[r] \arrow[d] & X_{K(n)} \arrow[d] \\ X_{E(n-1)} \arrow[r] & X_{K(n),E(n-1)} \end{tikzcd}$$

is a pullback

Cor:

$$X_{E(n)} \xrightarrow{\simeq} \underset{\ell \geq 1}{\underleftarrow{\mathrm{holim}}}\, X_{K(n_{i_1}),\dots,K(n_{i_\ell})}$$

$$n \geq n_{i_1} > n_{i_2} > \dots > n_{i_\ell} \geq 0$$

Lem: $BP_{K(n)} \simeq BP[v_n^{-1}]^{\wedge}_{(p,v_1,\dots,v_{n-1})}$

$$\underset{i_0,\dots,i_{n-1}}{\underleftarrow{\mathrm{holim}}}\, BP[v_n^{-1}] / (p^{i_0},\dots,v_{n-1}^{i_{n-1}})$$

Use: $\langle K(n)\rangle \underset{\text{Bousfield classes}}{\simeq} \langle BP[v_n^{-1}]/(p,\dots,v_{n-1})\rangle$

(pf)

<u>Need to show:</u>

(1) $BP[v_n^{-1}]^{\wedge}_{I_{n-1}}$ is $K(n)$-local

(2) $BP \longrightarrow BP[v_n^{-1}]^{\wedge}_{I_{n-1}}$ $K(n)$-equivalence

(1) follows from fact

$$\begin{tikzcd}
\dfrac{BP[v_n^{-1}]}{(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})} \arrow[d] & \Sigma^{(-)} \dfrac{BP[v_n^{-1}]}{(p, \ldots, v_{n-1})} \arrow[l] \\
\dfrac{BP[v_n^{-1}]}{(p^{i_0}, \ldots v_k^{i_k - 1}, \ldots v_{n-1}^{i_n - 1})} &
\end{tikzcd}$$

and $\operatorname{holim}(K(n)\text{-local}) = K(n)\text{-local}$

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(2) $K(n)_*$ (either)

$$K(n)_* BP = \left(\frac{E(n)}{(v_1, \ldots, v_{n-1})}\right)_* (-)$$

$$E(n)_* BP = E(n)_*[t_1, t_2, \ldots]$$

$$\frac{BP_*[v_n^{-1}]}{(v_1, \ldots, v_{n-1})}[t_1, t_2, \ldots \ldots]$$

Compute:

$$BP_{E(n)} \simeq \underleftarrow{\mathrm{holim}}\; BP_{K(n_1), \ldots, K(n_\ell)}$$

Spectral sequence

$$H^{s,t}\left(\bigoplus_{i=0}^{n} \pi_* BP_{K(i)} \to \bigoplus_{i=0}^{n} \pi_* BP_{K(i), K(j)} \to \cdots \to \pi_* BP_{K(0), K(1), \ldots, K(n)}\right) \Rightarrow \pi_{t-s} BP_{E(n)}$$

$$\lim{}^* \pi_* BP_{K(n_1), \ldots, K(n_\ell)} \Rightarrow \pi_* \underleftarrow{\mathrm{holim}}\; BP_{K(n_i), \ldots}$$

$\wr$

"local cohomology" can remove completions

$$H^*\left(\bigoplus_{i=0}^{n} BP_*[v_i^{-1}] \to \bigoplus_{i=0}^{n+1} BP_*[v_i^{-1}, v_j^{-1}] \to \cdots \to BP_*[v_1^{-1}, v_2^{-1}, \ldots, v_n^{-1}]\right)$$

Only set

$$H^0 = BP_*$$

$$H^{n+1} = \frac{BP_*[v_1^{-1}, \dots, v_n^{-1}]}{\bigoplus_i BP_*[p^{-1}, \dots, \hat{v}_i^{-1}, \dots, v_n^{-1}]}$$

$$= \frac{BP_*}{(p^\infty, v_1^\infty, \dots, v_n^\infty)}$$

$$= \operatorname{colim}_{I=(i_0,\dots,i_n)} \Sigma^{-\|I\|} \frac{BP_*}{(p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n})}$$

$\|I\| = \sum_j 2 i_j (p^j - 1)$

generated by

$$\text{monomial} \to \frac{x}{p^{i_0} v_1^{i_1} \cdots v_n^{i_n}} \qquad i_j > 0$$

"Laurent series"

e.g. $\mathbb{Z}/p^\infty = \varinjlim \mathbb{Z}/p^n = \mathbb{Q}/\mathbb{Z}$

$\frac{a}{p^n} \qquad p \nmid a$

Thm:

$$\pi_* BP_{E(n)} = BP_* \oplus \Sigma^{-n-1} \frac{BP_*}{(p^\infty, v_1^\infty, \dots, v_n^\infty)}$$

Note: v_n is not invertible in $\pi_* BP_{E(n)}$!

$M_n BP$:

$$M_n BP \simeq M_n BP_{K(n)}$$

$$\pi_* BP_{K(n)} \longrightarrow \pi_* BP_{K(n), E(n-1)} \longrightarrow \pi_* \Sigma M_n BP$$

$$\pi_* BP_{K(n), E(n-1)} \cong \pi_* BP_{K(n)} \oplus \Sigma^{-n} \underbrace{\frac{\pi_* BP_{K(n)}}{(p^\infty, \dots, v_{n-1}^\infty)}}_{\frac{BP_*[v_n^{-1}]}{(p^\infty, \dots, v_{n-1}^\infty)}}$$

$$\Longrightarrow \quad \pi_* M_n BP \cong \Sigma^{-n-1} \frac{BP_*[v_n^{-1}]}{(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty)}$$

Get AH css;

$$\mathrm{Ext}^{s,t}_{BP_*BP}\left(\frac{BP_*[v_n^{-1}]}{(p^\infty, \dots, v_{n-1}^\infty)}\right) \Rightarrow \mathrm{Ext}^{s+n,t}_{BP_*BP}(BP_*)$$

Thm

$$M_n S \simeq \operatorname*{colim}_{\substack{\longrightarrow \\ I=(i_0,\dots,i_{n-1})}} \Sigma^{-\|I\|-n-1} M(p^{i_0}, \dots, p^{i_{n-1}})_{E(n)}$$

(pf) $\Sigma^{-\|I\|-n-1} M(I)$ is a finite cx w/ top cell in dim 0 type n.

$$\begin{array}{ccc}
\Sigma^{-\|I\|-n-1} M(I)_{E(n)} & \xrightarrow{(*)} & M_n S^0 \\
\| & & \downarrow \\
\Sigma^{-\|I\|-n-1} M(I)_{E(n)} & \longrightarrow & S^0_{E(n)} \\
\downarrow & & \downarrow \\
* & \longrightarrow & S_{E(n-1)}
\end{array}$$

Both are $E(n)$-local

Suffice to prove $(*)$ is an $E(n)$-equivalence

$E(n)$ Landweber exact

$\Longrightarrow$ suffices to prove $(*)$ is BP-equivalence

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Periodicity in chromatic layers

$$\begin{tikzcd}
x \in \pi_* S_{(p)} \arrow[r] & \pi_* S_{E(n)} \arrow[r] & \pi_* S_{E(n-1)} \\
\tilde{x} \in \pi_* M_n S \arrow[ur] & & \\
\tilde{\tilde{x}} \in \pi_* \Sigma^{-\|I\|-n-1} M(I)_{E(n)} \arrow[u] \arrow[dr, "v_n^{i_n k}"] & & \\
& \pi_{*+2k i_n(p^n-1)} \Sigma^{-\|I\|-n-1} M(I)_{E(n)} \arrow[r] & \pi_{*+2k(p^n-1)} S_{E(n)}
\end{tikzcd}$$

$x \mapsto \neq 0$ in $\pi_* S_{E(n)}$, $\mapsto 0$ in $\pi_* S_{E(n-1)}$

$\tilde{\tilde{x}} \mapsto \tilde{\tilde{x}}_k \in \pi_{*+2k i_n(p^n-1)} \Sigma^{-\|I\|-n-1} M(I)_{E(n)}$, $\quad x_k \in \pi_{*+2k(p^n-1)} S_{E(n)}$

$\{x_k\}_{k\in\mathbb{Z}} = \infty$ chromatic family in $\pi_* S_{E(n)}$

period $= 2 i_n (p^n - 1)$

$x_0 = x$

Warning (1) x_k may be zero!

(though $\tilde{\tilde{x}}_k$ never zero since x_n^{in} is an $E(n)$-equiv.)

(2)

$$\begin{tikzcd} \pi_* S_{(p)} \arrow[d] & x \arrow[d, mapsto] & \exists ?\, x_k \arrow[d, mapsto] \\ \pi_* S_{E(n)} & x & x_k \end{tikzcd}$$

not necc.

e.g. $k \ll 0$, this would produce elements in negative degrees

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Telescope conj:

$\Longrightarrow$ x_k comes from $\pi_* S_{(p)}$ for $k \gg 0$

<u>Conversely</u> $\longleftrightarrow$ Suppose for any $I = (i_0, \dots, i_{n-1})$, $\exists$ lift

$$\begin{array}{ccc} \pi_* \Sigma^{-\|I\|-n-1} M(I) & \longrightarrow & \pi_* S_{(p)} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \tilde{x} & & x \end{array}$$

Suppose $\tilde{x}$ is v_n-periodic

i.e. $v_n^{i_n k} \tilde{x} \neq 0$ $\forall$ k

"x is v_n-periodic"

<u>Tel conj</u>

$$\Rightarrow \quad \begin{array}{c} \tilde{x} \neq 0 \\ \uparrow \\ \Sigma^{-\|I\|-n-1} M(I)_{E(n)} \end{array}$$

$\Rightarrow$ chrom filt $(x) \geq n$

So Telescope conj links

(chrom filt) $\longleftrightarrow$ (v_n-periodicity in $\pi_* S_{(p)}$)