TOPOLOGY AND ECONOMICS

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1. Introduction

In the first part of this article we will work through the notions of smooth manifolds, tangent spaces and differentiation on manifolds (all in the simple Euclidean setting) to prove an important result in differential topology, Sard's Theorem. Our exposition is entirely based on the first three sections of Milnor's lovely little book *Topology from the Differentiable Viewpoint*.

In the second part of this article, we will study a simple but nevertheless insightful model of economy with tools of topology. In particular, we will put the two powerful theorems of topology, Brouwer's Fixed Point Theorem and Sard's Theorem, into attractive uses.

2. Differential Topology in Euclidean Space

2.1. Smooth Map and Manifolds.

Definition 2.1.1. Let U be an open subset in $\mathbb{R}^k$, and let Y be an arbitrary subset of $\mathbb{R}^l$. The map $f : U \to Y$ is *smooth* if at every point in U partial derivatives of f of all order exist and are continuous.

Definition 2.1.2. Let X and Y be arbitrary subsets of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively. The map $f : X \to Y$ is *smooth* if at every point $x \in X$, there exist an open set $U \subset \mathbb{R}^k$ containing x and a *smooth extension* (smooth in the sense of Definition 2.1.1) $F : U \to \mathbb{R}^l$ such that F agrees with f in $U \cap X$.

It is clear that Definition 2.1.2 is consistent with Definition 2.1.1.

We first note that smooth maps are continuous. Suppose the mapping $f : X \to Y$ is smooth; take any $x \in X$, the smooth extension of f at x, $F : U \to \mathbb{R}^l$ is continuous by construction; but then f agrees with a continuous function (F) on a neighborhood ($U \cap X$) of x in X, thus f must be continuous at x.

The smoothness of a map is also preserved by restriction to subsets. Let f defined above be again smooth, and let G be an arbitrary subset

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of X. Then $f|G$ is smooth: for any $x \in G$, let $F : U \to \mathbb{R}^l$ be a smooth extension of f at x; we see immediately that F is also a smooth extension of $f|G$ at x.

The smoothness of maps is also preserved by composition:

Proposition 2.1.1. *Suppose $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^l$, $Z \subset \mathbb{R}^j$, $f : X \to Y$ and $g : Y \to Z$ are smooth. Then $g \circ f : X \to Z$ is smooth.*

Proof. Fix an $x \in X$; let $y = f(x) \in Y$. There exist smooth extensions $F : U \to \mathbb{R}^l$ and $G : V \to \mathbb{R}^j$ where U and V are neighborhoods (open in $\mathbb{R}^k$ and in $\mathbb{R}^l$, respectively) of x and y, respectively. Let $U' = F^{-1}(V) \subset U$; clearly U' is a neighborhood of x in R^k. Now we can easily check that $G \circ F|U' : U' \to \mathbb{R}^j$ is a smooth extension of $g \circ f$ at x. $\square$

Definition 2.1.3. A map $f : X \to Y$ is called a *diffeomorphism* if f is bijective and if both f and f^{-1} is smooth.

Clearly, a diffeomorphism is also a homeomorphism. Also, given $G \subset X$, $f|G : G \to f(G)$ is a diffeomorphism if $f : X \to Y$ is a diffeomorphism.

Definition 2.1.4. A subset $M \subset \mathbb{R}^k$ is called a *smooth manifold* of *dimension* m if each $x \in M$ has a neighborhood W in M that is diffeomorphic to an open subset U of $\mathbb{R}^m$.

Any particular diffeomorphism $g : U \to W$ is called a *parametrization* of W. The inverse diffeomorphism $g^{-1} : W \to U$ is called a system of *coordinates* on W.

It follows from the definition that any discrete set in some Euclidean space (for example, a finite set) is a smooth manifold of dimension 0 (by definition, $\mathbb{R}^0 = \{0\}$).

It is easily seen that for a manifold M, if N is an open subset of M, then N also a manifold (i.e. a *submanifold* of M), with $\dim N = \dim M$ open subset. Since $\mathbb{R}^k$ is trivially a manifold of dimension k, we see that any of its open subsets is a manifold of dimension k.

2.2. Tangent Spaces and Derivatives.

Definition 2.2.1. For an open subset U of $\mathbb{R}^k$, its *tangent space* at any $x \in U$, TU_x, is defined to be the entire vector space $\mathbb{R}^k$. Thus in this sense manifolds are a generalization of open sets.

Definition 2.2.2. For a smooth map $f : U \to Y \subset \mathbb{R}^l$, where U is open in $\mathbb{R}^k$, its *derivative* at any $x \in U$, $df_x : \mathbb{R}^k \to \mathbb{R}^l$, is defined as $df_x(h) = \lim_{t \to 0}(f(x + th) - f(x))/t$, for $h \in \mathbb{R}^k$.

From multivariable calculus, we learned that the derivative above takes the form $df_x(h) = Jh$, where J is the $l \times k$ Jacobian matrix of partial derivatives. Also from multivariable calculus, we have the following proposition:

Proposition 2.2.1 (Chain Rule)**.** *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$ be open sets and $W \subset \mathbb{R}^j$ an arbitrary set, and let $f : U \to V$ and $g : V \to W$ be smooth maps. Then for any $x \in U$, $d(g \circ f)_x = dg_y \circ df_x$, where $y = f(x)$.*

And the following intuitive fact which we have suspected all along follows easily from the chain rule:

Proposition 2.2.2. *If f is a diffeomorphism between open sets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$, then k must equal l, and the linear map $df_x : \mathbb{R}^k \to \mathbb{R}^l$ must be nonsingular for all $x \in U$.*

Proof. Fix an $x \in U$ and let $y = f(x)$. The composition $f^{-1} \circ f$ is the identity map on U, thus $d(f^{-1} \circ f)_x = d(f^{-1})_y \circ df_x = i_{\mathbb{R}^k}$, where $i_{\mathbb{R}^k}$ is the identity map on $\mathbb{R}^k$. Likewise we have $df_x \circ d(f^{-1})_y = i_{\mathbb{R}^l}$, which immediately implies our conclusion. □

A partial converse to the above proposition is the following famous theorem:

Theorem 2.2.3 (Inverse Function Theorem)**.** *Let $f : U \to \mathbb{R}^k$ be a smooth map, with U open in $\mathbb{R}^k$, and let x in a point in U. If the derivative $df_x : \mathbb{R}^k \to \mathbb{R}^k$ is nonsingular, then there exists an open set U' in U containing x such that f maps U' diffeomorphically onto $f(U')$ and $f(U')$ is open in $\mathbb{R}^k$*

Definition 2.2.3. For a smooth manifold $M \subset \mathbb{R}^k$ of dimension m and any point $x \in M$, the *tangent space* of M at x, $T_x(M)$, is defined as follows: choose a parametrization $g : U \to M$ of a neighborhood $g(U)$ of x in M, where U is open in $\mathbb{R}^m$; then $T_x(M) = dg_u(\mathbb{R}^m)$, where $g(u) = x$ and the derivative dg_u is in the sense of Definition 2.2.2.

The first thing we must do is to show that Definition 2.2.3 is well-defined:

Proposition 2.2.4. *TM_x in Definition 2.2.3 is independent of the parametrization $g : U \to M$.*

Proof. Let $h : V \to M$ be another parametrization of another neighborhood $h(V)$ of x in M (as usual, V is open in $\mathbb{R}^m$), and let $v = h^{-1}(x)$. Our $T_x(M)$ will be well-defined if $dg_u(\mathbb{R}^m) = dh_v(\mathbb{R}^m)$.

Let $W = g(U) \cap h(V)$, $U' = g^{-1}(W)$ and $V' = h^{-1}(W)$. Clearly U' and V' are open in $\mathbb{R}^m$. Note that $h^{-1} \circ g : U' \to V'$ is a diffeomorphism, and that we have the following diagram:

$$\begin{tikzcd} U' \arrow[rr, "h^{-1}\circ g"] \arrow[dr, "g"'] & & V' \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

We then take derivatives and apply the chain rule (Proposition 2.2.1) to get:

$$\begin{tikzcd} \mathbb{R}^m \arrow[rr, "d(h^{-1}\circ g)_u"] \arrow[dr, "dg_u"'] & & \mathbb{R}^m \arrow[dl, "dh_v"] \\ & \mathbb{R}^k & \end{tikzcd}$$

Since $h^{-1} \circ g$ is a diffeomorphism, $d(h^{-1} \circ g)_u$ is an isomorphism by Proposition 2.2.2. Then from the above diagram we conclude that $dg_u(\mathbb{R}^m) = dh_v(\mathbb{R}^m)$. $\square$

Proposition 2.2.5. *Let $M \subset \mathbb{R}^k$ be a m-dimensional smooth manifold. Then for all $x \in M$, $T_x(M)$ is a m-dimensional vector subspace of $\mathbb{R}^k$.*

Proof. Fix an $x \in M$. We see immediately from Definition 2.2.3 that $T_x(M)$ is a subspace of $\mathbb{R}^k$. Let $g : U \to g(U) \subset M$ be a parametrization of $g(U) \ni x$, where U is open in $\mathbb{R}^m$ and $g(U)$ open in M; let $u = g^{-1}(x)$. Since the inverse $g^{-1} : g(U) \to U$ is a smooth function, there exists a smooth extension $F : W \to \mathbb{R}^m$ (where W is open in $\mathbb{R}^k$) that agrees with g^{-1} on $W \cap g(U)$. Let $U' = g^{-1}(W \cap g(U))$; clearly U' is open in $\mathbb{R}$. Then we have the following diagram:

$$\begin{tikzcd} U' \arrow[rr, "\text{inclusion}"] \arrow[dr, "g"'] & & \mathbb{R}^m \\ & W \arrow[ur, "F"'] & \end{tikzcd}$$

We again take derivatives and apply the chain rule to get:

$$\begin{tikzcd} \mathbb{R}^m \arrow[rr, "\text{identity}"] \arrow[dr, "dg_u"'] & & \mathbb{R}^m \\ & \mathbb{R}^k \arrow[ur, "dF_x"'] & \end{tikzcd}$$

The diagram implies that dg_u is injective, which means that $T_x(M) = dg_u(\mathbb{R}^m)$ is of dimension m. $\square$

We now generalize the definition of derivative to smooth maps between manifolds:

Definition 2.2.4. Given two smooth manifolds $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$, of dimension m and n respectively, and a smooth map $f : M \to N$; for any $x \in M$ let $y = f(x) \in N$. The derivative of f, $df_x : T_x(M) \to T_y(N)$, is defined as follows. Let $F : W \to \mathbb{R}^l$ be the smooth extension of f at x, where W contains x and is open in $\mathbb{R}^k$. Then for each $v \in T_x(M)$, define $df_x(v)$ to be equal to $dF_x(v)$, where $dF_x : \mathbb{R}^k \to \mathbb{R}^l$ is defined in the sense of Definition 2.2.2.

Proposition 2.2.6. *df_x in Definition 2.2.4 maps $T_x(M)$ into $T_y(N)$ and is independent of $F : W \to \mathbb{R}^l$.*

Proof. We first choose parametrization $g : U \to M \subset \mathbb{R}^k$ and $h : V \to N \subset \mathbb{R}^l$ for neighborhoods $g(U)$ of x and $h(V)$ of y; notice that U is open in $\mathbb{R}^m$ and V open in $\mathbb{R}^n$; let $u = g^{-1}(x)$ and $v = h^{-1}(y)$. Let $W' = f^{-1}(h(V)) \cap W$ and $U' = f^{-1}(W')$; since W' is open in M, $U' \subset U$ is open in $\mathbb{R}^m$; and clearly, $x \in g(U')$. For convenience, we will rename U' to U and $g|U'$ to g. Then we have $g(U) \subset W$ and $f(g(U)) \subset h(V)$. Thus, we have the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbb{R}^l \\ U \arrow[u, "g"] \arrow[r, "h^{-1}\circ f\circ g"] & V \arrow[u, "h"'] \end{tikzcd}$$

Taking derivatives and applying the chain rule, we have

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^l \\ \mathbb{R}^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1}\circ f\circ g)_u"] & \mathbb{R}^n \arrow[u, "dh_v"'] \end{tikzcd}$$

Now it is clear that dF_x maps $T_x(M) = dg_u(\mathbb{R}^m)$ into $T_y(N) = dh_v(\mathbb{R}^n)$; thus so does df_x. Additionally, we see that for any $w \in T_x(M)$, we have $df_x(w) = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ (dg_u)^{-1}(w)$ (simply go around the bottom of the diagram, note that it does not matter whether or not dg_u is injective), which is independent of the smooth extension $F : W \to \mathbb{R}^l$ of f at x. $\square$

We can generalize the chain rule to manifolds as follows:

Proposition 2.2.7 (Chain Rule on Manifolds)**.** *Let $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$, $P \subset \mathbb{R}^j$ be smooth manifolds. If $f : M \to N$ and $g : N \to P$ are smooth, with $x \in M$, $y = f(x)$, then $d(g \circ f)_x = dg_y \circ df_x$.*

Proof. First choose a smooth extension $G : V \to \mathbb{R}^j$ of g at y such that $V \ni y$ is open in $\mathbb{R}^l$. Then choose a smooth extension $F : U \to \mathbb{R}^l$ of f at x such that $U \ni x$ is open in $\mathbb{R}^k$ and $F(U) \subset V$ (c.f. the proof of Proposition 2.1.1). Then, for any $v \in T_x(M)$, $df_x(v) = dF_x(v)$ and

$d(g \circ f)_x(v) = d(G \circ F)_x(v)$, while for any $w \in T_y(N)$, $dg_y(w) = dG_y(w)$. Our conclusion follows since by the chain rule for open sets we have $d(G \circ F)_x(v) = dG_y \circ dF_x(v)$. □

Proposition 2.2.8. *Let $M \subset N \subset \mathbb{R}^k$ where M and N are smooth manifolds. Then we have $T_x(M) \subset T_x(N) \subset \mathbb{R}^k$ for all $x \in M$.*

Proof. Let $i : M \to N$ be the inclusion map. Then for any $x \in M$, $di_x : T_x(M) \to T_x(N)$ is also an inclusion map. □

And we have the following proposition whose proof is completely analogous to that of Proposition 2.2.2.

Proposition 2.2.9. *Let M and N be two smooth manifolds in some Euclidean spaces. If $f : M \to N$ is a diffeomorphism, then for any $x \in M$ with $y = f(x)$, $df_x : T_x(M) \to T_y(N)$ is an isomorphism of vector spaces. In particular, the dimension of M must be equal to the dimension of N.*

2.3. **Sard's Theorem.**

Theorem 2.3.1 (Sard's Theorem)**.** *Let $f : U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^m$, and let $C = \{x \in U \mid rank\ df_x < n\}$. Then, the image $f(C) \subset \mathbb{R}^n$ has Lebesgue measure zero.*

The proof of Sard's Theorem is rather involved and deserves a treatment in its own section. But let us first explore the various implications and generalizations of Sard's Theorem.

Since a set of Lebesgue measure 0 (called a *null set*) in $\mathbb{R}^m$ cannot contain any nonempty open set of $\mathbb{R}^m$, the complement of a null set of $\mathbb{R}^m$ is dense in $\mathbb{R}^m$.

Notice that in the statement of Sard's theorem $m < n$ means that $C = U$; thus in this case the theorem simply says that $f(U) \subset$ is null in $\mathbb{R}^m$, which is intuitive but nevertheless non-trivial.

Definition 2.3.1. Given a smooth m-dimensional manifold M, a subset $R \subset M$ is of *measure 0* (or *null*) in M if for any $x \in R$ and any parametrization $g : U \to M$ of a neighborhood $g(U)$ of x, $g^{-1}(g(U) \cap R)$ is of measure 0 in $\mathbb{R}^m$.

Proposition 2.3.2. *Suppose that M is a m-dimensional smooth manifold, and that $R \subset M$ is null in M, then $M - R$ is dense in M.*

Proof. Suppose $M - R$ is not dense in M, then there exists a $x \in R$ and $V \ni x$ open in M such that $V \subset R$. Let $g : U \to M$ be a parametrization of a neighborhood $g(U)$ of x, where U is open in $\mathbb{R}^m$. Then $g^{-1}(g(U) \cap V)$ is open in $\mathbb{R}^m$ and is nonempty (it contains

$g^{-1}(x)$), which implies that it is not null in $\mathbb{R}^m$. But this means that $g^{-1}(g(U) \cap R)$ which contains $g^{-1}(g(U) \cap V)$ is also not null in $\mathbb{R}^m$. $\square$

Definition 2.3.2. For a smooth $f : M \to N$ from a manifold of dimension m to a manifold of dimension n, let C be the set of all $x \in M$ such that the derivative at x $df_x : T_x(M) \to T_{f(x)}(N)$ has rank less than n (i.e. not surjective). Then C is called the set of *critical points* of f, while $f(C)$ the set of *critical values*. Likewise, $M - C$ is called the set of *regular points* of f, while $N - f(C)$ the set of *regular values*.

Theorem 2.3.3 (Sard's Theorem for Manifolds)**.** *The set of critical values of a smooth map $f : M \to N$ between manifolds is of measure* 0 *in N.*

Proof. Suppose that N is of dimension n and M of dimension m. Let $K \subset N$ be the set of critical values of f. For any $y \in K$ and any parametrization $h : V \to N$ of a neighborhood $h(V)$ of y, we will show that $h^{-1}(h(V) \cap K)$ is null in $\mathbb{R}^n$.

For each $x \in f^{-1}(h(V) \cap K) \subset M$, choose a parametrization $g_x : U_x \to M$ of a neighborhood $g_x(U_x)$ of x such that $f(g_x(U_x)) \subset h(V)$. Let $C_x \subset U_x$ be the set of critical points of the smooth map $h^{-1} \circ f \circ g_x : U_x \to \mathbb{R}^n$.

Since M is imbedded in some Euclidean space, we can choose a countable subset $I \subset f^{-1}(h(V) \cap K)$ such that

$$f^{-1}(h(V) \cap K) \subset \bigcup_{x \in I} g_x(U_x)$$

Apply $h^{-1} \circ f$ to both sides of the containment above, we get (remember that K is the image of f over the set of critical points of f):

$$h^{-1}(h(V) \cap K) \subset \bigcup_{x \in I} h^{-1} \circ f \circ g_x(U_x)$$

Thus for any $z \in h^{-1}(h(V) \cap K)$, $z \in h^{-1} \circ f \circ g_x(U_x)$ for some $x \in I$. But then for that $x \in I$, $z \in h^{-1} \circ f \circ g_x(C_x)$. Thus,

$$h^{-1}(h(V) \cap K) \subset \bigcup_{x \in I} h^{-1} \circ f \circ g_x(C_x)$$

Applying Sard's Theorem to each $h^{-1} \circ f \circ g_x$ ($x \in I$), we conclude that $h^{-1}(h(V) \cap K)$ is of measure 0 in $\mathbb{R}^n$. $\square$

Theorem 2.3.4 (Preimage Theorem)**.** *If $f : M \to N$ is a smooth map between manifolds of dimensions $m \geq n$, and if $y \in N$ is a nontrivial regular value (i.e. $f^{-1}(y) \neq \emptyset$), then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension dimension $m - n$.*

Proof. Fix an $x \in f^{-1}(y)$. Since y is a regular value, the rank of $df_x : T_x(M) \to T_y(N)$ is n, and the null space $\ker df_x \subset T_x(M)$ is an $(m-n)$-dimensional vector space.

Suppose that $M \subset \mathbb{R}^k$, choose a linear map $L : \mathbb{R}^k \to \mathbb{R}^{m-n}$ that is nonsingular on the subspace $\ker df_x \subset T_x(M)$. Now define $F : M \to N \times \mathbb{R}^{m-n}$ by $F(z) = (f(z), L(z))$ for $z \in M$. The derivative $dF_x : T_x(M) \to T_y(N) \times \mathbb{R}^{m-n}$ is clearly given by $dF_x(v) = (df_x(v), L(v))$ for any $v \in T_x(M)$.

Thus, dF_x is of rank m. Applying the Inverse Function Theorem for manifolds, we have F maps $U \ni x$ open in M diffeomorphically onto $V \ni (y, L(x))$ open in $N \times \mathbb{R}^{m-n}$. Then, F maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times \mathbb{R}^{m-n}) \cap V$, which is open in $y \times \mathbb{R}^{m-n}$. But $y \times \mathbb{R}^{m-n}$ is diffeomorphic to $\mathbb{R}^{m-n}$ with the natural diffeomorphism $\pi : y \times \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$, therefore $f^{-1}(y) \cap U$ is the neighborhood of x in $f^{-1}(y)$ that is diffeomorphic to the open set $\pi((y \times \mathbb{R}^{m-n}) \cap V)$ of $\mathbb{R}^{m-n}$. □

2.4. **Proof of Sard's Theorem.** Our proof of Sard's Theorem follows §3 of Milnor in an almost verbatim manner. In particular, we will prove Sard's Theorem by doing induction on n and m. Note that the theorem makes sense for $m \geq 0$ and $n \geq 1$. The theorem is obviously true when $m = 0$ for all $n \geq 1$.

As in the statement of the theorem, let C be the set of f's critical points. And let $C_i \subset C$ be the set of all $x \in U$ such that all partial derivatives of f of order $\leq i$ vanish at x. Then we have a descending sequence of sets

$$C \supset C_1 \supset C_2 \supset C_3 \supset \dots$$

Our proof consists of three steps:

Step 1: $f(C - C_1)$ has measure zero in $\mathbb{R}^n$. When $n = 1$, we have $C = C_1$, so there is nothing to prove in this step.

We now assume that $n \geq 2$.

We will need the following version of Fubini's Theorem:

Theorem 2.4.1 (Fubini)**.** *Let A be a measurable set in $\mathbb{R}^n$. Then A is of measure 0 in $\mathbb{R}^n$ if $A \cap (t \times \mathbb{R}^{n-1})$ is of measure 0 in $t \times \mathbb{R}^{n-1}$ for all $t \in \mathbb{R}$.*

For each $\bar{x} \in C - C_1$ we will find a neighborhood $V \ni x$ open in $\mathbb{R}^m$ such that $f(V \cap C)$ is of measure 0 in $\mathbb{R}^n$. Since $C - C_1$ can be covered by countably many of these neighborhoods, this will imply that $f(C - C_1)$ is of measure 0 in $\mathbb{R}^n$.

Since $\bar{x} \notin C_1$, there exist integers $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $\partial f_i/\partial x_j$ does not vanish at $\bar{x}$. Without loss of generosity we can assume that $i = j = 1$.

We define the map $h : U \to \mathbb{R}^m$ by $h(x) = (f_1(x), x_2, \ldots, x_m)$ for all $x \in U$. By construction $dh_{\bar{x}}$ is nonsingular, thus h maps some neighborhood V of $\bar{x}$ in U diffeomorphicaly onto a set V' open in $\mathbb{R}^m$. Let $g = f \circ h^{-1}$; then g maps V' into $\mathbb{R}^n$. Note that by the chain rule the set C' of critical points of g is precisely $h(V \cap C)$. Thus the set of critical values of g is $f(V \cap C)$.

For each $(t, x_2, \ldots, x_m) \in V'$, we have $g(t, x_2, \ldots, x_m) \in t \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, i.e. g maps hyperplanes to hyperplanes. Thus, for a fixed $t \in \mathbb{R}$ let $\pi_k : t \times \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ be the natural homeomorphic projection and let $V'_t = \pi_m((t \times \mathbb{R}^{m-1}) \cap V')$; notice that V'_t is open in $\mathbb{R}^{m-1}$ and possibly empty. We derive a map $g^t : V'_t \to \mathbb{R}^{n-1}$ from g in the following manner: for $z \in V'_t$, let $g^t(z) = \pi_n(g(t \times z))$.

We can easily check that $y \in \mathbb{R}^{n-1}$ is a critical value of g^t if and only if $t \times y$ is a critical value of g. Thus, the π_n projection of the intersection between the set of critical values of g (the set $g(C')$) and $t \times \mathbb{R}^{n-1}$ is equal to the set of critical value of g^t. But by our induction hypothesis applied to g^t, the set of critical value of g^t is of measure zero in $\mathbb{R}^{n-1}$. Therefore by Fubini's Theorem we conclude that $g(C')$ is of measure 0 in $\mathbb{R}^n$.

Step 2: $f(C_i - C_{i+1})$ is of measure 0 in $\mathbb{R}^n$ for $i \geq 1$. For each $\bar{x} \in C_i - C_{i+1}$ there is some $(i+1)$th derivative $\partial^{i+1} f_r/\partial_{s_1} \ldots \partial_{s_{i+1}}$ that is not zero at $\bar{x}$. Let $w(x) = \partial^i f_r/\partial_{s_2} \ldots \partial_{s_{i+1}}$; then $w(\bar{x}) = 0$ but $\partial w/\partial x_{s_1}(\bar{x}) \neq 0$. Without loss of generosity we assume that $s_1 = 1$. Then the map $h : U \to \mathbb{R}^m$ defined by $h(x) = (w(x), x_2, \ldots, x_m)$ carries some neighborhood V of $\bar{x}$ diffeomorphically onto an open set V' of $\mathbb{R}^m$.

Note that h carries $C_i \cap V$ into the hyperplane $0 \times \mathbb{R}^{m-1}$. Again we let $g = f \circ h^{-1} : V' \to \mathbb{R}^n$. Let $\tilde{g} : \pi_m((0 \times \mathbb{R}^{m-1}) \cap V') \subset \mathbb{R}^{m-1} \to \mathbb{R}^n$ be the induced map from g analogous to the g^t in the previous case. Applying our induction hypothesis to $\tilde{g}$, we conclude that the set of critical values of $\tilde{g}$ is of measure 0 in $\mathbb{R}^n$. But $\pi_m(h(C_i \cap V))$ are certainly critical points of $\tilde{g}$, thus $f(C_i \cap V) = g(h(C_i \cap V)) = \tilde{g}(\pi_m(h(C_i \cap V)))$ is of measure 0 in $\mathbb{R}^n$. Since $C_i - C_{i+1}$ is covered by countably many such sets V, we conclude that $f(C_i - C_{i+1})$ has measure 0 in $\mathbb{R}^n$.

Step 3: $f(C_k)$ is of measure 0 in $\mathbb{R}^n$ when k is sufficiently large. Let $I^m \subset U$ be a cube with edge δ. It suffices to show that when k is sufficiently large, $f(C_k \cap I^m)$ has measure 0 in $\mathbb{R}^n$, since C_k can be covered by countably many of these $C_k \cap I^m$.

From Taylor's theorem, the compactness of I^m, and the definition of C_k, we have

$$f(x+h) = f(x) + R(x,h)$$

where

$$||R(x,h)|| \leq c||h||^{k+1}$$

and $x \in C_k \cap I^m$, $x+h \in I^m$, and c is a constant that only depends on f and I^m.

We now subdivide I^m into r^m cubes of edge δ/r. Let I_1 be one of these cubes that contains a point $x \in C_k$. Then, any point of I_1 can be written as $x+h$, with $||h|| \leq \sqrt{m}\delta/r$.

Thus, $f(I_1)$ lies in a cube of edge a/r^{k+1} centered about $f(x)$, where $a = 2c(\sqrt{m}\delta)^{k+1}$. Hence $f(C_k \cap I^m)$ is contained in a union of at most r^m cubes having total volume

$$V \leq r^m(\frac{a}{r^{k+1}})^n = a^n r^{m-(k+1)n}$$

Clearly, for k such that $k+1 > m/n$, V tends to zero as r tends to infinity. Therefore, for $k > m/n - 1$, $f(C_k \cap I^m)$ is of measure zero in $\mathbb{R}^n$.

3. Theory of General Equilibrium

General equilibrium is a branch of economic theory that studies the equilibrium (i.e. when supply equals demands for every goods in the economy) state of an economy inhabited by agents with different and sometimes conflicting interests. An important question one asks in general equilibrium is that for a given model of economy, does an equilibrium exist? And if so, how many equilibria are there and what sort of local and global properties do they possess? If this section, we will attempt to answer these questions for a simple model of economy where agents consume by exchanging their initially owned goods in a "competitive" market.

A word on our notation: we let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \text{ for } i = 1, \ldots, n\}$ and $\mathbb{R}^n_{++} = \{(x_1, \ldots, x_n) \mid x_i > 0 \text{ for } i = 1, \ldots, n\}$. For $x, y \in \mathbb{R}^n$, we write $x \geq y$ when $x_i \geq y_i$ for $i = 1, \ldots, n$.

3.1. **Model of Exchange Economy.**

Our economy consists of I consumers ($i = 1, \ldots, I$) and L consumption goods ($l = 1, \ldots, L$). A bundle of consumption goods is represented by a point in the goods space $\mathbb{R}^L_+$. Each consumption good has a market price, thus the space of all market prices is also $\mathbb{R}^L_+$.

We assume that each consumer i has an initial endowment which is a bundle of consumer goods represented by $\omega^i \in \mathbb{R}^L$. Given the current market prices for consumer goods $p \in \mathbb{R}^L$, the "net worth" of this endowment ω^i is $p \cdot \omega^i$; this will be the income of consumer i. Notice that here we have assumed that each consumer is able to sell all consumer goods in the market according to the current price.

Finally, consumer i might not be entirely happy with his endowment ω^i. Therefore he would like to buy consumer goods in the market with his income derived from his endowment; that is, he would sell his endowment in the market, and with the money generated from his sale he would buy a bundle of goods with which he would be most satisfied. This behavior is captured by his demand function $f^i : \mathbb{R}^L_+ \times (0, \infty) \to \mathbb{R}^L_+$; that is, given the current market price p and his income $p \cdot \omega^i$, the demand function gives us a bundle of goods $f^i(p, p \cdot \omega_i)$ that consumer i would buy and consume.

We will let the set of all possible endowment bundles of all I consumers to be the space $(\mathbb{R}^L_+)^I = \mathbb{R}^{IL}_+$. Then for each $\omega \in \mathbb{R}^{IL}_+$, ω^i denotes the endowment bundle of consumer i, i.e. $(\omega_{(i-1)L+1}, \ldots, \omega_{iL}) \in \mathbb{R}^L_{++}$, which is consistent with what we used before.

Definition 3.1.1. A state of economy characterized by $\omega \in \mathbb{R}^{IL}_+$ and $p \in \mathbb{R}^L_+$ is in *general equilibrium* if

$$\sum_{i=1}^{I} f^i(p, p \cdot \omega^i) = \sum_{i=1}^{I} \omega^i$$

We usually fix an $\omega \in \mathbb{R}^{IL}_+$ and look at the set of price vectors that induce general equilibrium.

The condition general equilibrium simply says that the total demand of each good is equal to its total supply. Intuitively, a state of economy not in general equilibrium seems unstable, since some people have unfulfilled desire, and should converge (via adjusting the price vector $p \in \mathbb{R}^L_+$) to a general equilibrium state. This intuition thus justifies our focus on general equilibrium.

Finally, we define the aggregate excess demand function $Z : \mathbb{R}^L_+ \times \mathbb{R}^{IL}_+ \to \mathbb{R}^L$ by

$$Z(p, \omega) = \sum_{i=1}^{I} (f^i(p, p \cdot \omega^i) - \omega^i)$$

Clearly, an economy is in general equilibrium if and only if $Z(p, \omega) = 0$.

For the existence of an equilibrium price vector, we need the following assumptions on the demand functions and their derived aggregate excess demand:

Assumption 3.1. Each $f^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+$ $(i = 1, \ldots, I)$ satisfies:

(1) f^i is continuous.
(2) $f^i(cp, cw) = f^i(p, w)$ for all $c \in (0, \infty)$, $p \in \mathbb{R}^L_+$ and $w \in \mathbb{R}_+$.
(3) $f^i(p, w) \cdot p = w$ for all $p \in \mathbb{R}^L_+$ and $w \in \mathbb{R}_+$.
$Z : \mathbb{R}^L_+ \times \mathbb{R}^{IL}_+ \to \mathbb{R}^L$ satisfies:
(4) Given an $p \in \mathbb{R}^L_+$ and an $\omega \in \mathbb{R}^{IL}_+$, for $l = 1, \ldots, L$, $p_l = 0$ implies $Z_l(p, \omega) > 0$.

Part (1) of Assumption 3.1 is needed for technical (topological) reasons and seems relatively harmless; after all, people's behaviors usually do not change radically given a small change in price and income. Part (2) says that multiplying the price and his income by a constant factor will not affect agent i's consuming decision. Part (3) says that each consumer will use up all of his income for consumption and will not waste a dime. Part (4) simply says that everyone wants free goods. These assumptions seem to describe the consumer-behavior of a typical materialistic individual in a capitalist economy, and thus seem not too implausible or restrictive. We will show in the next subsection that although each person in our model economy is "greedy" and acts only in self-interest (as dictated by his demand function), there still exists a set of prices that leaves no desire unfulfilled and no resource wasted.

3.2. **Existence of Price Equilibrium.** Throughout this subsection we assume that Assumption 3.1 on the demand functions holds.

Lemma 3.2.1. *For any $\omega \in \mathbb{R}^{IL}_+$ and any $p \in \mathbb{R}^L_+$, if $Z(p, \omega) \leq 0$, then $Z(p, \omega) = 0$.*

Proof. Suppose that $Z_l(p, \omega) < 0$ for some l. Then $p_l = 0$: since $p \in \mathbb{R}^L_+$, $p_l \neq 0$ means $Z_l(p, \omega)p_l < 0$, which implies that $Z(p, \omega) \cdot p < 0$; but by Assumption 3.1(3) we have $p \cdot Z(p, \omega) = 0$. However, $p_l = 0$ implies that $Z_l(p, \omega) > 0$ by Assumption 3.1(4). □

Lemma 3.2.2. *Suppose that $\omega \in \mathbb{R}^{IL}_+$ and $p \in \mathbb{R}^L_+$ induce general equilibrium, then $p \in \mathbb{R}^L_{++}$.*

Proof. This follows immediately from Assumption 3.1(4). □

We will focus our attention on the price simplex $S^{L-1}_+ = \{p \in \mathbb{R}^L_+ \mid p_1 + \ldots + p_L = 1\}$. Let $S^{L-1}_{++} = S^{L-1}_+ \cap \mathbb{R}^L_{++}$.

Theorem 3.2.3 (Existence of General Equilibrium)**.** *For any initial endowment $\omega \in \mathbb{R}^{IL}_+$, there exists a price vector $p \in S^{L-1}_{++} \subset \mathbb{R}^L_{++}$ that induces general equilibrium.*

Proof. Fix an $\omega \in \mathbb{R}_+^{IL}$.

We define a map $g : S_+^{L-1} \to S_+^{L-1}$ by

$$g_l(p) = \frac{p_l + \max(0, Z_l(p,\omega))}{1 + \sum_{l=1}^{L} \max(0, Z_l(p,\omega))} \text{ for } l = 1, \dots, L$$

Clearly, g is continuous, so Brouwer's Fixed Point Theorem applies, and we have $g(p^*) = p^*$ for some $p^* \in S_+^{L-1}$.

Then, we have for $l = 1, \dots, L$

$$p_l^* \sum_{l=1}^{L} \max(0, Z_l(p^*, \omega)) = \max(0, Z_l(p^*, \omega))$$

Multiplying by $Z_l(p^*, \omega)$ and summing over $l = 1, \dots, L$, we arrive at

$$\left(\sum_{l=1}^{L} p_l^* Z_l(p^*, \omega)\right)\left(\sum_{l=1}^{L} \max(0, Z_l(p^*, \omega))\right) = \sum_{l=1}^{L} Z_l(p^*, \omega) \max(0, Z_l(p^*, \omega))$$

Since $Z(p^*, \omega) \cdot p^* = 0$, we conclude that $Z(p^*, \omega) \leq 0$ since each term in the summation on the RHS is non-negative. We then apply Lemma 3.2.1 and Lemma 3.2.2 to p^* and ω to obtain $Z(p^*, \omega) = 0$ and $p^* \in S_{++}^{L-1}$. □

It is clear that without Assumption 3.1(4), we can only show that $Z(p^*, \omega) \leq 0$, i.e. given the price vector p^* people's demands are sustainable but there are possibilities of unwanted (and thus free) goods.

3.3. **Local Uniqueness of Equilibria.** Things would be great if for a given endowment distribution there exists an unique price vector that induces general equilibrium. In this case our model makes sharp prediction on the future state of the economy and therefore can be tested against real economic data. However, only with some unrealistically strong assumptions could we prove uniqueness of price equilibrium. Therefore, we will hope for the next best thing: each price equilibrium is somehow nicely separated from the other equilibria (so that there is the possibility of some non-equilibrium states of economy converging to one of them), and they are not too many of them. We will tackle the first question in this section, using the theory of smooth manifolds and Sard's Theorem developed earlier in this article. With additional tools of differential topology, one can show that the number of price equilibria is finite and in fact is always *odd* (see Nagata, §4 and 9). Although there seems not to be any economic interpretation for this oddness of equilibria, it does spare us the effort to prove the existence of equilibrium (i.e. no need to prove Theorem 3.2.3) since zero is not an odd number!

For a technical reason that will become apparent later, we will assume that both our endowment space and price space are strictly positive; that is, $\omega \in \mathbb{R}^{IL}_{++}$ and $p \in \mathbb{R}^{L}_{++}$. Correspondingly, our demand functions are now defined only on $\mathbb{R}^{L}_{++} \times \mathbb{R}_{++}$. Additionally, in this subsection we assume that they satisfy the following conditions:

Assumption 3.2. Each $f^i : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_{+}$ $(i = 1, \ldots, I)$ satisfies:

(1) f^i is smooth.
(2) $f^i(cp, cw) = f^i(p, w)$ for all $c \in \mathbb{R}_{++}$, $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}_{++}$.
(3) $f^i(p, w) \cdot p = w$ for all $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}_{++}$.

The aggregate excess demand function $Z : \mathbb{R}^L_{++} \times \mathbb{R}^{IL}_{++} \to \mathbb{R}^L$ is still defined as before. For a given $\omega \in \mathbb{R}^{IL}_{++}$, let $E(\omega) = \{p \in S^{L-1}_{++} \mid Z(p, \omega) = 0\}$, i.e. the set of price equilibria associated with ω. We restrict our attention to S^{L-1}_{++} because by Assumption 3.2(2) if $Z(p, \omega) = 0$ then $Z(cp, \omega) = 0$; but p and cp are really be the same equilibrium, since they differ only in "monetary units."

Definition 3.3.1. Equilibria in $E(\omega)$ are *locally unique* if $E(\omega)$ is discrete in S^{L-1}_{++}.

We define the map $F : S^{L-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{(I-1)L}_{++} \to \mathbb{R}^{IL}$ by

$$F(p, w^1, \omega^2, \ldots, \omega^I) = (f^1(p, w^1) + \sum_{i=2}^{I} f^i(p, p \cdot \omega^i) - \sum_{i=2}^{I} \omega^i, \omega^2, \ldots, \omega^I)$$

Note that F is a smooth map between two manifolds of dimension LI.

Proposition 3.3.1. *Given an $\omega \in \mathbb{R}^{IL}_{++}$, π maps $F^{-1}(\omega)$ homeomorphically onto $E(\omega)$, where π is the projection of the first L coordinates.*

Proof. Fixed an $\omega \in \mathbb{R}^{IL}_{++}$. Suppose $(p, w^1, \omega^2, \ldots, \omega^I) \in F^{-1}(\omega)$ for some $p \in S^{L-1}_{++}$ and $w^1 \in \mathbb{R}_{++}$, then by the construction of F we have

$$f^1(p, w^1) + \sum_{i=2}^{I} f^i(p, p \cdot \omega^i) - \sum_{i=2}^{I} \omega^i = \omega^1$$

Multiplying by p, using Assumption 3.2(3) and doing some subtractions we arrive at $w^1 = p \cdot \omega^1$. Plug this back to the equation above, we conclude that $p \in E(\omega)$.

Now suppose that we have $p \in E(\omega)$. Let $w^1 = p \cdot \omega^1$, then we have $(p, w^1, \omega^2, \ldots, \omega^I) \in F^{-1}(\omega)$. $\square$

Thus, to see if $E(\omega)$ is locally unique, we only need to check if $F^{-1}(\omega)$ is discrete in $S^{L-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{(I-1)L}_{++}$; this is when the machinery developed in the first part of this article becomes handy:

Theorem 3.3.2 (Local Uniqueness of Price Equilibria)**.** *The set of $\omega \in \mathbb{R}^{IL}_{++}$ such that $E(\omega)$ is locally unique is dense in $\mathbb{R}^{IL}_{++}$.*

Proof. Let $\omega \in \mathbb{R}^{IL}$ be a regular value of F. Then, $F^{-1}(\omega)$ is either an zero-dimensional manifold or empty by the Pre-image Theorem. For either case $F^{-1}(\omega)$ is discrete in $S^{L-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{(I-1)L}_{++}$.

By Sard's Theorem, the set of regular values $\omega \in \mathbb{R}^{IL}$ of F is dense in $\mathbb{R}^{IL}$. Thus, its intersection with $\mathbb{R}^{IL}_{++}$ is dense in $\mathbb{R}^{IL}_{++}$. $\square$

3.4. **Conclusion.** If our demand functions satisfy Assumptions 3.1, and their restrictions to $\mathbb{R}^{L}_{++} \times \mathbb{R}_{++}$ also satisfy Assumption 3.2, then for almost every initial endowment distribution $\omega \in \mathbb{R}^{IL}_{++}$, its set of price equilibria $E(\omega)$ is *both* non-empty (i.e. an equilibrium exists) and locally unique (i.e. equilibria are nicely separated, topologically).

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