HW4 Solutions

Note Title 3/2/2009

7.2(2) By the n^{th} term test,

$$a_n \longrightarrow 0$$

$$\Longrightarrow \quad a_n < 1 \quad \text{for } n \gg 0$$

sequence location thm (say $n \geq N$)

Therefore, for $n \geq N$:

$$0 \leq a_n^2 \leq a_n$$

because $a_n < 1$

$\Longrightarrow$ (comparison thm) $\sum_{n \geq N} a_n^2$ converges

$\Longrightarrow$ (tail convergence thm) $\sum a_n$ converges

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7.4(1)

(a) $$0 \leq \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

Since $\sum \frac{1}{n^{3/2}}$ converges (p-series) $\Longrightarrow$ (comparison thm) $\sum \frac{\sqrt{n}}{n^2+1}$ converges

(b)

$$\frac{\left|\frac{(n+1)^2}{2^{n+1}}\right|}{\left|\frac{n^2}{2^n}\right|} = \frac{(n+1)^2}{2n^2} = \frac{n^2+2n+1}{2n^2}$$

$$= \frac{1}{2} + \frac{1}{n} + \frac{1}{2n^2} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$$

Ratio test $\Longrightarrow$ $\sum \frac{n^2}{2^n}$ converges

(c)

$$0 \le \left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges $\underset{\text{comparison test}}{\Longrightarrow}$ $\sum \left|\frac{\cos n}{n^2}\right|$ converges,

$\underset{\text{absolute convergence thm}}{\Longrightarrow}$ $\sum \frac{\cos n}{n^2}$ converges

(d)

$$\frac{\left|\frac{((n+1)!)^2}{(2(n+1))!}\right|}{\frac{(n!)^2}{(2n)!}} = \frac{(n+1)^2}{(2n+1)(2n+2)}$$

$$\underset{\text{multiply top + bottom by } 1/n^2}{=} \frac{(1+\frac{1}{n})(1+\frac{1}{n})}{(2+1/n)(2+2/n)} \xrightarrow{\text{algebraic limit thms}} \frac{1}{2} < 1$$

By ratio test, $\sum_i \frac{(n!)^2}{(2n)!}$ converges.

$\boxed{7.5(4)}$ Note that since $\lim \frac{|a_n|}{|b_n|} = 1$

$$\Rightarrow \lim \frac{|b_n|}{|a_n|} = 1$$

Therefore, the statement of the asymptotic comparison test is symmetric in $\{a_n\}$ and $\{b_n\}$

Therefore, it suffices to prove one direction of the implication. We will prove

$$\left(\sum_i |a_n| \text{ converges}\right) \Rightarrow \left(\sum_i |b_n| \text{ converges}\right)$$

Suppose $\sum_i |a_n|$ converges

Since $\frac{|a_n|}{|b_n|} \to 1$, For $\varepsilon = 1/2$

$$\frac{|a_n|}{|b_n|} \approx_{1/2} 1 \quad \text{for } n \gg 0.$$

therefore:

$$\frac{1}{2} < \frac{|a_n|}{|b_n|} < \frac{3}{2} \quad \text{for } n \gg 0$$

$$\Longrightarrow \quad \frac{1}{2}|b_n| < |a_n| < \frac{3}{2}|b_n| \quad \text{for } n \gg 0$$

By comparison, we deduce that

$\sum \frac{1}{2}|b_n|$ converges

Therefore $\sum |b_n| = 2 \cdot \sum \frac{1}{2}|b_n|$ converges

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$\boxed{7.6\ (1a\text{-}b)}$

$\{n^{1/3}\}$ increasing $\Rightarrow \{\frac{1}{n^{1/3}}\}$ decreasing

(a) $\sum \frac{(-1)^n}{n^{1/3}}$

Since $n^{1/3} \longrightarrow \infty$

$\frac{1}{n^{1/3}} \longrightarrow 0$ (thm $S.L(\infty)$)

$\Longrightarrow$ Cauchy's thm $\sum \frac{(-1)^n}{n^{1/3}}$ converges

$\sum \left|\frac{(-1)^n}{n^{1/3}}\right| = \sum \frac{1}{n^{1/3}}$ p-series with $p = 1/3 \Rightarrow$ diverges

So $\sum \frac{(-1)^n}{n^{1/3}}$ converges conditionally.

(b) $\sum_1^\infty \left|\frac{(-1)^n}{n^2+1}\right| = \sum_1^\infty \frac{1}{n^2+1}$.

$$0 \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

Since $\sum_1^\infty \frac{1}{n^2}$ converges,

$\underset{\text{comparison thm}}{\Longrightarrow} \sum_1^\infty \frac{1}{n^2+1}$ converges

$\Longrightarrow \sum_1^\infty \frac{(-1)^n}{n^2+1}$ converges absolutely.