The equation for a plane

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This is a quick note to tell you how to easily write the equation of a plane in 3-space.

1 Planes passing through the origin

Planes are best identified with their normal vectors. Thus, given a vector $\mathbf{V} = \langle v_1, v_2, v_3 \rangle$, the plane P_0 that passes through the origin and is perpendicular to $\mathbf{V}$ is the set of all points (x, y, z) such that the position vector $\mathbf{X} = \langle x, y, z \rangle$ is perpendicular to $\mathbf{V}$. In other words, we have

$$\langle x, y, z \rangle \cdot \mathbf{V} = v_1 x + v_2 y + v_3 z = 0$$

so the equation for the plane P_0 is $v_1 x + v_2 y + v_3 z = 0$.

2 Planes passing through any point

That was only a plane through the origin! We want equations for all planes, including the ones that don't pass through the origin. How do we specify such planes? Given a vector $\mathbf{V}$, there are *infinitely* many planes perpendicular to $\mathbf{V}$. Think about why this is true – for instance, the planes perpendicular to $\hat{\mathbf{k}}$ are those of the form $z = C$ for some constant C. Since C can be any real number, there are infinitely many such planes.

To specify one of these planes, we just need to specify a point that it passes through. This should be compared to lines in 2-space: there are infinitely many lines with slope m, but only one line with slope m that passes through a given point (x_0, y_0). Suppose that P is the plane passing through (x_0, y_0, z_0) which is perpendicular to $\mathbf{V}$. What is the equation for P? If we subtract the position vector $X_0 = \langle x_0, y_0, z_0 \rangle$ from all of the points in the plane, this has the effect that the plane is translated to the origin. Call this translated plane P_0 – this was the sort of plane we dealt with in Section 1. Thus the point (x, y, z) with position vector $\mathbf{X} = \langle x, y, z \rangle$ is in P if and only if the translated point

$$\mathbf{X} - \mathbf{X_0}$$

is in P_0. This is equivalent to

$$(\mathbf{X} - \mathbf{X_0}) \cdot \mathbf{V} = 0$$

or

$$\mathbf{X} \cdot \mathbf{V} = \mathbf{X_0} \cdot \mathbf{V}.$$

when expanded out, this equation reads

$$v_1 x + v_2 y + v_3 z = \mathbf{X_0} \cdot \mathbf{V}$$

and this is the equation of the plane P which is perpendicular to $\mathbf{V}$ and passes through (x_0, y_0, z_0).

3 Half spaces

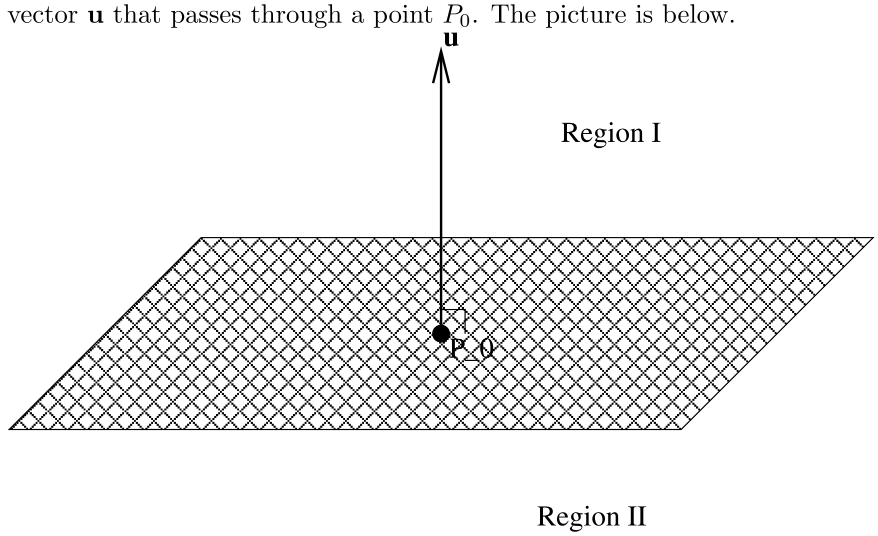
A plane P as above divides 3-space into two regions. These regions are given by the inequalities

$$v_1 x + v_2 y + v_3 z > \mathbf{X_0} \cdot \mathbf{V} \tag{1}$$
$$v_1 x + v_2 y + v_3 z < \mathbf{X_0} \cdot \mathbf{V} \tag{2}$$

The region given by Equation 1 is the region that the vector $\mathbf{V}$ is pointing *toward*, and the region given by Equation 2 is the region that the vector $\mathbf{V}$ is pointing away from. Try this out (draw) with a simple example to see what I mean by this.

4 A simplified approach

As an afterthought to writing this document I realized the following approach may be simpler to conceptualize. Consider the plane perpendicular to a fixed vector $\mathbf{u}$ that passes through a point P_0. The picture is below.



The plane divides 3-space into two regions: region I, which lies on the same side of the plane as $\mathbf{u}$, and region II, which lies on the other side of the plane. For a point P in 3-space, we have

- P lies on the plane if and only if $\vec{P_0P} \cdot \mathbf{u} = 0$.
- P lies in region I if and only if $\vec{P_0P} \cdot \mathbf{u} > 0$.
- P lies in region II if and only if $\vec{P_0P} \cdot \mathbf{u} < 0$.

This formulation should be intuitively more clear, and more useful for problem 4 of part B.