Logic Seminar

UNIVERSITY OF NOTRE DAME
Department of Mathematics

Speaker: Philip Ehrlich
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Thursday, April 9, 2009
2:00 PM
258 Hurley Hall (Note room change)

NOTE: Room Change

Title: Surreal Numbers, Conway Names, and the Simplicity Hierarchy

Abstract:

J. H. Conway's real-closed field of *surreal numbers* [1] contains the reals and the ordinals as well as a great many other numbers including , , , and to name only a few. Indeed, this particular real-closed field, which Conway calls , is so remarkably inclusive that, subject to the proviso that numbers -- construed here as members of ordered "number" fields -- be individually definable in terms of sets of von Neumann-Bernays-Gödel set theory with Global Choice, it may be said to contain "All Numbers Great and Small." However, in addition to its distinguished structure as an ordered field has a rich algebraico-tree-theoretic structure, or *simplicity hierarchy*, that emerges from the recursive clauses in terms of which it is defined [2].

Among the striking algebraico-tree-theoretic surreal features is that ordinal exponentiation with base extends to an operation in such a way that every surreal number can be expressed uniquely as a generalized power series of the form -- called the *Conway name* (or *normal form*) of [2] -- where is an ordinal, is a strictly decreasing series of surreals, and is a series of nonzero real numbers. Indeed, every such expression is the Conway name of some surreal number, the Conway name of an ordinal being just its Cantor normal form [2].

In the present paper some of the relations that exist between the Conway names of surreal numbers and the simplicity hierarchy are explored. Among the questions that are answered are the following two that are motivated by *No*'s structure as a *lexicographically ordered full binary tree* (*of height On*):

(i) Given the Conway name of a surreal number *x*, what are the Conway names of its two immediate successors?

(ii) Given a chain of surreal numbers of infinite limit length, what is the Conway name of the immediate successor of the chain?

Since every real-closed ordered field is isomorphic to an *initial subfield* of *No* [2]—a subfield of *No* that is an initial subtree of *No*—the answers to these questions not only shed light on the recursive unfolding of *No*, but on the recursive unfolding in *No* of real-closed ordered fields more generally. In the final portion of the paper some preliminary remarks on this matter will be provided.

[1] J.H. Conway, *On Numbers and Games*, Academic Press, 1976.

[2] P. Ehrlich, "Number systems with simplicity hierarchies: A generalization of Conway's theory of surreal numbers", *The Journal of Symbolic Logic* 66, pp. 1231-1258.