Complex Analysis Candidacy (Fall, 2004)

To complete the examination, you will work a total of 6 problems, choosing three from among 1-5, and three from among 6-10. If you turn in solutions to more than three problems from either section, only the first three will be graded.

Part A. Choose any three from among Problems 1-5 below.

1. (a) Define what it means for a function to be complex differentiable at a point.

(b) At what points $z \in \mathbf{C}$ is the function $f(z) = |z|^2$ complex differentiable? Justify your answer.

2. Let $f(z) = \frac{1}{1-z^2}$.

(a) What is the Laurent series for f on the set $\{|z| < 1\}$?

(b) What is the Laurent series for f on the set $\{|z| > 1\}$?

3. (a) State Liouville's Theorem.

(b) Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function, and let c be a real constant satisfying $|f(z) - f(w)| \le c|z - w|$ for all $z, w \in \mathbf{C}$. Show that f is affine (i.e. $f(z) = Az + B$ for some constants $A, B \in \mathbf{C}$).

4. (a) State the Cauchy Integral Formula (there are several variations on this–any of them will serve).

(b) Give a proof of the fundamental theorem of algebra that uses complex analysis.

5. Evaluate

$$\int_0^\infty \frac{x^2}{1+x^4}\,dx \ .$$

Part B. Choose any three from among Problems 6-10 below.

6. Is there a holomorphic function $f(z)$, defined in a neighborhood of the origin, such that $f(z)^2 = z$? Justify your answer.

7. (a) State the Riemann mapping theorem.

(b) Give a sequence of conformal mappings whose composition sends the unit disk bijectively onto the Riemann sphere minus the interval $[-1, 1]$.

8. (a) State the maximum principle.

(b) Let $a_1, \ldots, a_n$ be n given points on the unit circle $C = \{z : |z| = 1\}$. Prove that there is $z_0 \in C$ such that the sum of the distances from z_0 to the a_j is greater than or equal to n.

9. Let $f : \mathbf{C} \to \mathbf{C}$ be an injective holomorphic function. By analyzing the nature of the singularity at infinity, show that f is of the form $f(z) = az + b$.

10. Suppose $f(x + iy) = u(x, y) + iv(x, y)$ is analytic, and the real part $u(x, y)$ is a polynomial. Show that f is also a polynomial.