Reliable Computation of Codimension-1 and Codimension-2 Bifurcations Using Interval Analysis

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Outline

- Motivating Example
- Bifurcation Types and Test Functions
- Interval Newton/Generalized Bisection
- Results

Predator/Prey Systems

- Systems of ordinary differential equations that describe the rates of change in species biomass
- Model parameters have real-life, physical meaning
- These models can exhibit rich mathematical behavior including varying numbers and stability of equilibria

Example Model

$$\frac{dx_1}{dt} = x_1 r\left(1 - \frac{x_1}{K}\right) - \frac{a_2 x_1 x_2}{b_2 + x_1}$$

Tri-trophic model with a hyperbolic predator and a sigmoidal superpredator

$$\frac{dx_2}{dt} = e_2 \frac{a_2 x_1 x_2}{b_2 + x_1} - \frac{a_3 x_2^2 x_3}{b_3 + x_2^2} - d_2 x_2$$

x_1: prey
x_2: predator
x_3: superpredator

$$\frac{dx_3}{dt} = e_3 \frac{a_3 x_2^2 x_3}{b_3 + x_2^2} - d_3 x_3$$

r : prey growth rate
K : prey carrying capacity

a_i : max predation rate　b_i : half saturation constant
d_i : death rate　e_i : efficiency

Bifurcations of Equilibria

- Goal – locate equilibrium points and bifurcations in predator/prey models
- A bifurcation is a change in the topological type of the phase portrait as one or more system parameters are varied
  - One parameter allowed to vary – codimension-one bifurcation
  - Two parameters allowed to vary – codimension-two bifurcation
- Bifurcations are located by solving a nonlinear algebraic system consisting of the equilibrium conditions along with one or more augmenting (test) functions

Codim-1 Bifurcations and Test Functions

- Fold and transcritical bifurcations
  - Collision of equilibria that results in annihilation or an exchange in stability
  - The Jacobian, $J(\boldsymbol{x},\alpha)$, of the model has a single zero eigenvalue
  - Product of all eigenvalues must be zero: $\lambda_1\lambda_2\lambda_3 = 0$
  - Convenient test function (avoiding calculation of eigenvalues):

$$\det(J(\boldsymbol{x},\alpha)) = 0$$

Codim-1 Bifurcations and Test Functions

- Hopf bifurcation
  - $J(\boldsymbol{x},\alpha)$ has a pair of imaginary complex conjugate eigenvalues that cross the imaginary axis
  - Product of all possible pair sums must be zero:

    $$(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3) = 0$$
  - Convenient test function based on bialternate product

    $$\det\,(2J(\boldsymbol{x},\alpha) \otimes I) = 0$$
  - Can produce false-positives
  - Must screen solutions by checking if eigenvalues are imaginary conjugates

Codim-2 Bifurcations and Test Functions

- Fold-Fold: Two eigenvalues that are zero
- Fold-Hopf: One eigenvalue that is zero and a pair of purely imaginary complex conjugate eigenvalues
- Located by using both augmenting functions

$$\det(J(\boldsymbol{x},\alpha,\beta)) = 0$$

$$\det(2J(\boldsymbol{x},\alpha,\beta) \otimes I) = 0$$

Locating Bifurcations

- Must solve equilibrium conditions and augmenting function(s) for $\boldsymbol{x}$ and α (and β)
- Equation system may have multiple solutions
- Typically these systems are solved using a continuation-based strategy (e.g., Kuznetsov, 1991)
  - Initialization dependent
  - No guarantee of locating all branches
- Interval mathematics provides a method that is:
  - Initialization *independent*
  - Capable of locating *all* branches *with certainty*

Interval Newton/Generalized Bisection Method (IN/GB)

- Given a system of equations, an initial interval (bounds on all variables), and a solution tolerance:
  - IN/GB can find (enclose), *with mathematical and computational certainty*, *all* solutions to the equation system, or it can determine that no solutions exist (Moore, 1966; Kearfott, 1996)
  - The equation system must have a finite number of real roots in the initial interval
  - No strong assumptions or simplifications to the equation system are needed

IN/GB Method

Problem: Solve $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ for all roots in the interval $\mathbf{X}^{(0)}$

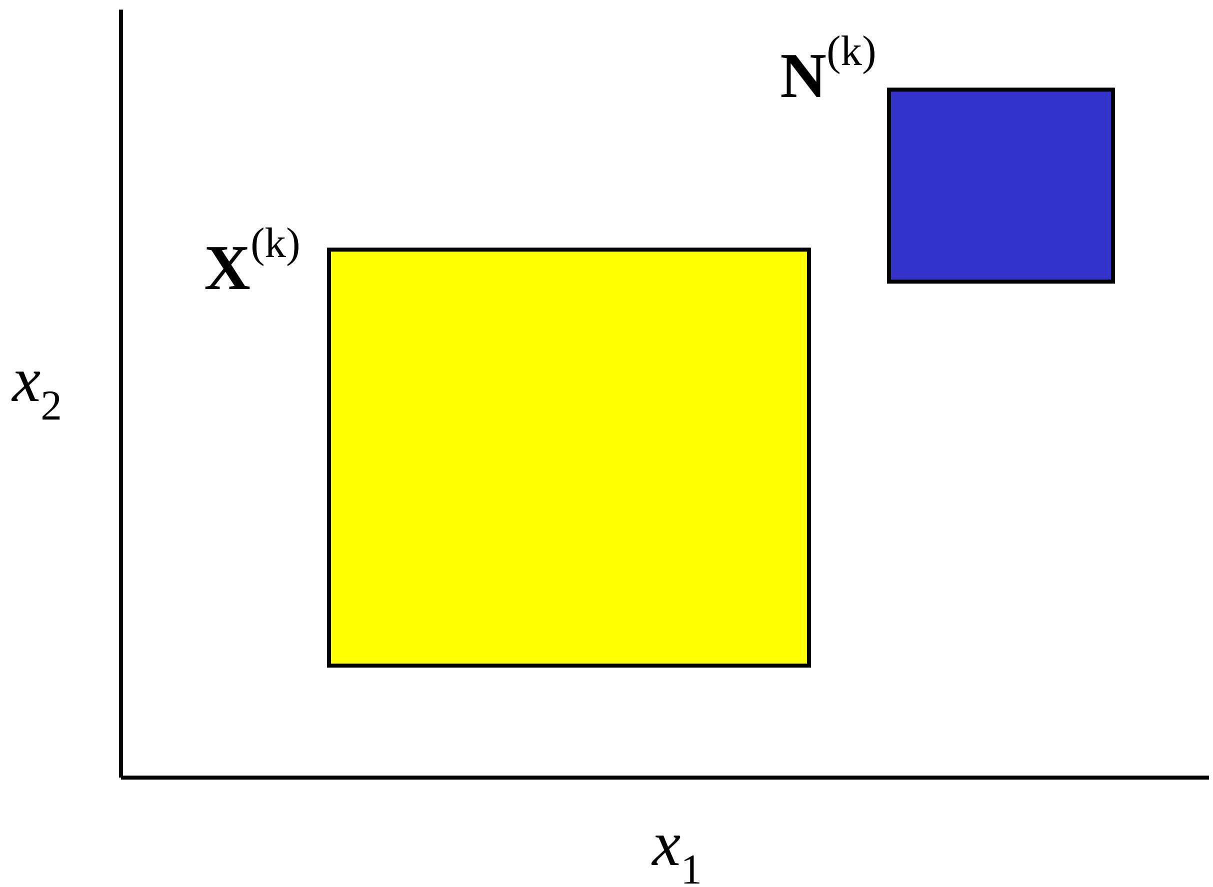
Basic iteration scheme: For a particular subinterval (box), $\mathbf{X}^{(k)}$, perform root inclusion test:

- Range test: Compute an interval extension (bounds on range) for each function in the system
  - If any element of the interval extension does not contain zero, delete the box. Otherwise continue.
- Interval Newton test: Compute the image, $\mathbf{N}^{(k)}$, of the box by solving the linear interval equation system

$$F'(\mathbf{X}^{(k)})\,(\mathbf{N}^{(k)} - \boldsymbol{x}^{(k)}) = -\boldsymbol{f}\,(\boldsymbol{x}^{(k)})$$

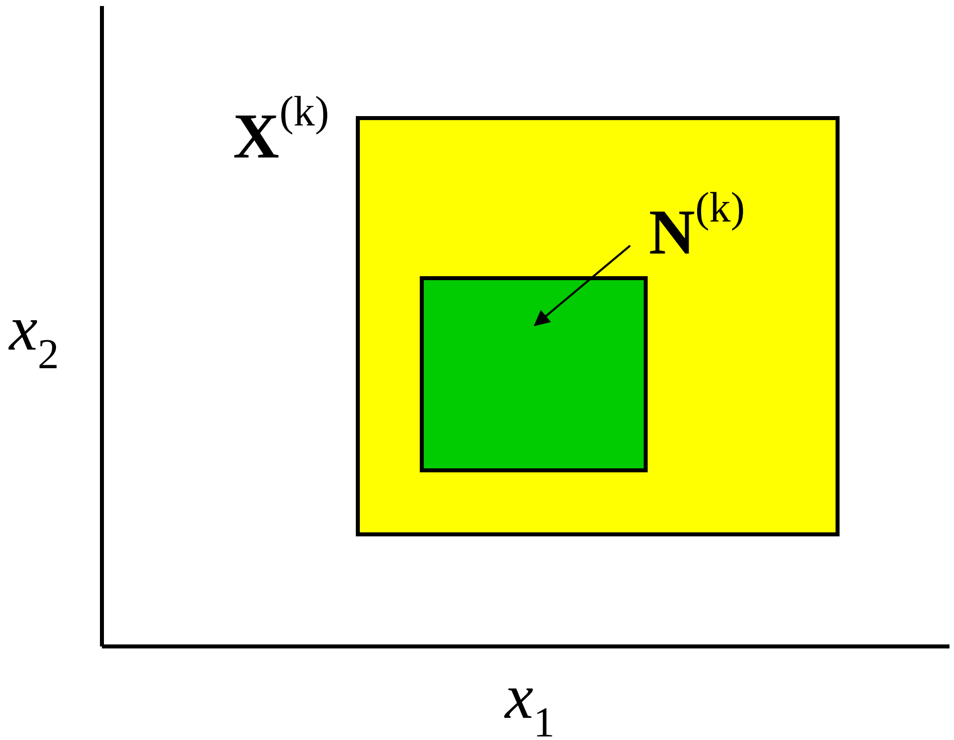
  - $\boldsymbol{x}^{(k)}$ is any point in $\mathbf{X}^{(k)}$
  - $F'(\mathbf{X}^{(k)})$ is the interval extension of the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$ over the interval $\mathbf{X}^{(k)}$

IN/GB Method: Interval Newton Test



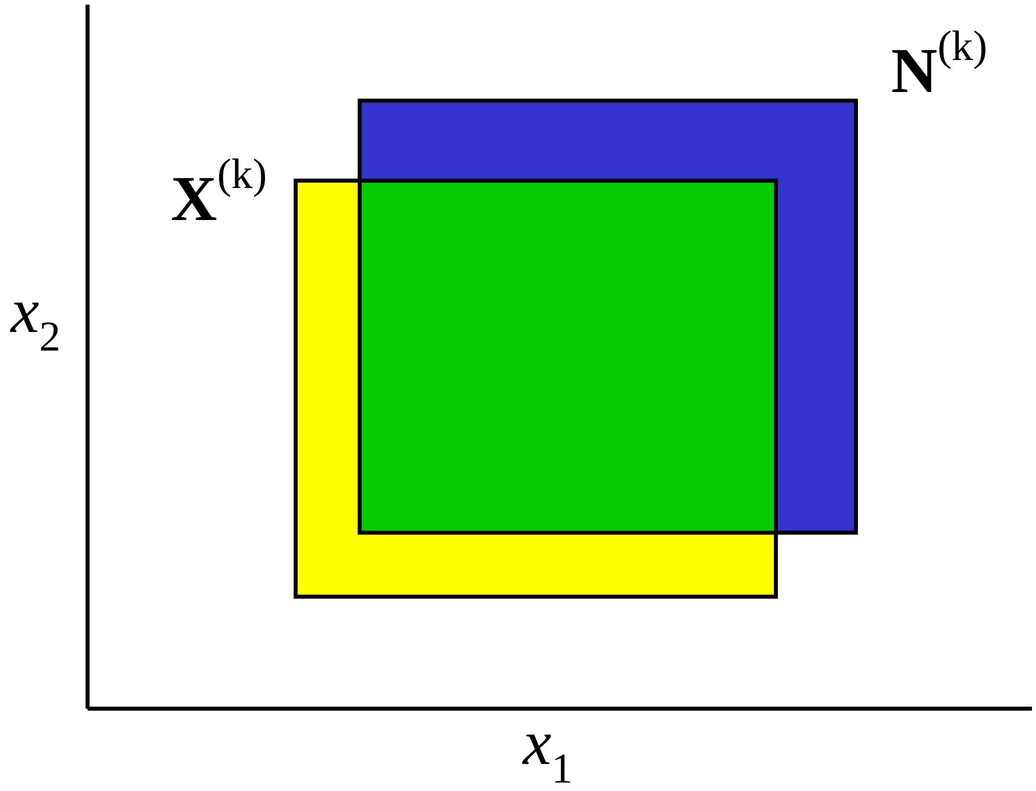
- There is no solution in $\mathbf{X}^{(\mathbf{k})}$

IN/GB Method: Interval Newton Test



- There is a *unique* solution in $\mathbf{X}^{(k)}$ that is also in $\mathbf{N}^{(k)}$
- Additional interval-Newton steps will tightly enclose the solution with quadratic convergence

IN/GB Method: Interval Newton Test



- Any solutions in $\mathbf{X}^{(\mathbf{k})}$ are in the intersection of $\mathbf{X}^{(\mathbf{k})}$ and $\mathbf{N}^{(\mathbf{k})}$
- If the intersection is sufficiently small, repeat the root inclusion test
- Otherwise, bisect the intersection and apply the root inclusion test to each resulting subinterval

Example Problems

- IN/GB was used to solve the equilibrium conditions and appropriate augmenting function(s) for $\boldsymbol{x}$ and the bifurcation parameter(s) of interest
- To generate codim-one bifurcation diagram (say r vs. K)
  - Set r, solve for bifurcation(s) with K as parameter
  - Increment r and repeat
  - Set K, solve for bifurcation(s) with r as parameter
  - Increment K and repeat

Rosenzweig-MacArthur Model

$$\frac{dx_1}{dt} = x_1 r\left(1 - \frac{x_1}{K}\right) - \frac{a_2 x_1 x_2}{b_2 + x_1}$$

Tri-trophic model with a hyperbolic predator and superpredator

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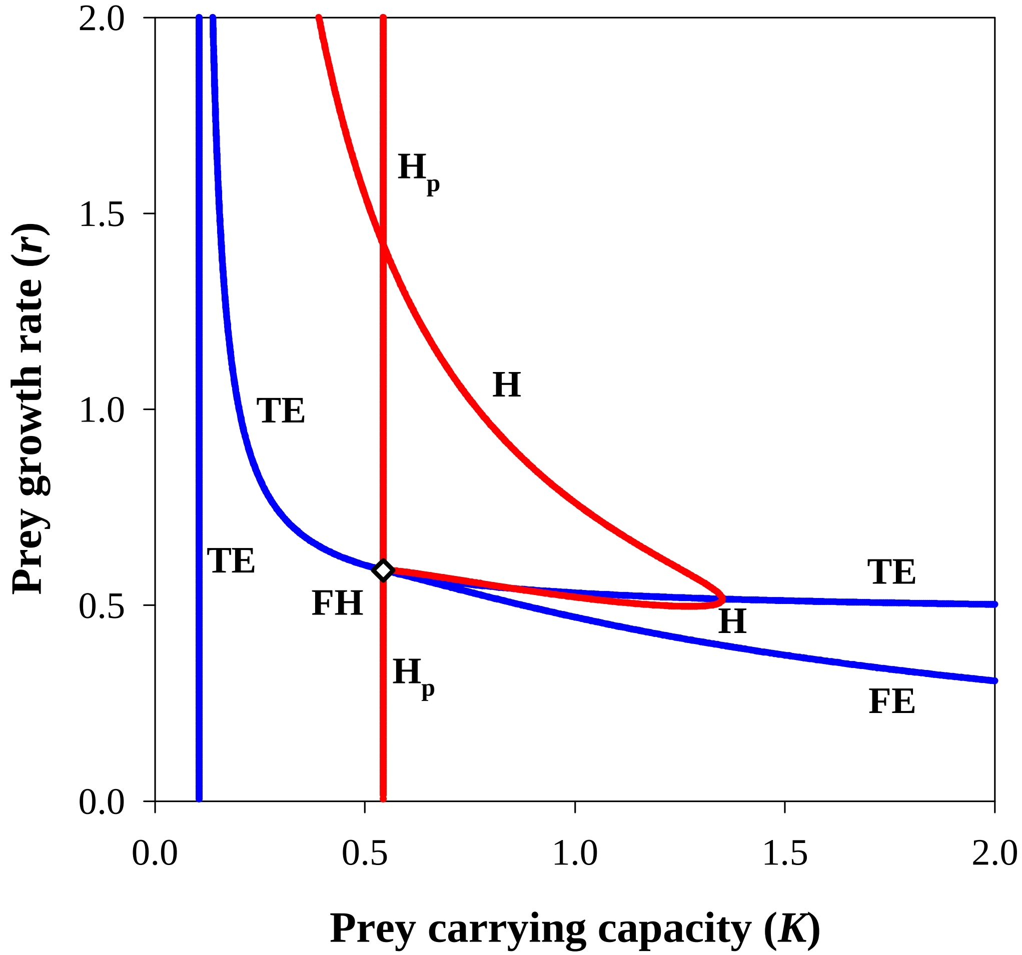
Rosenzweig-MacArthur Model
K vs. *r* Bifurcation Diagram



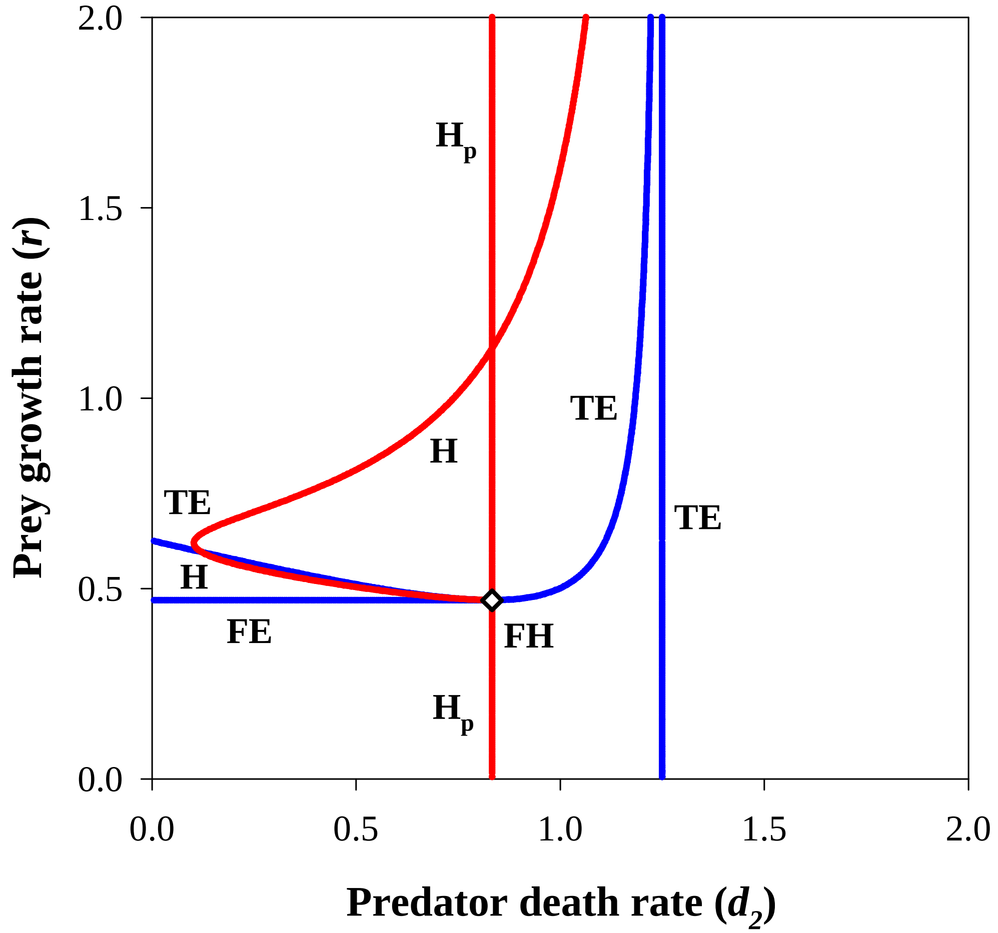
Bifurcation diagram matches results published in literature (Gragnani, 1998)

TE: Transcritical of Equilibria
FE: Fold of Equilibria

H: Hopf
H_p: Planar Hopf

FH: Fold-Hopf Codimension 2

Rosenzweig-MacArthur Model
d_2 vs. r Bifurcation Diagram



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Example Model

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Tri-trophic model with a hyperbolic predator and a sigmoidal superpredator

$$\frac{dx_2}{dt} = e_2 \frac{a_2 x_1 x_2}{b_2 + x_1} - \overbrace{\underbrace{\frac{a_3 x_2^2 x_3}{b_3 + x_2^2}}} - d_2 x_2$$

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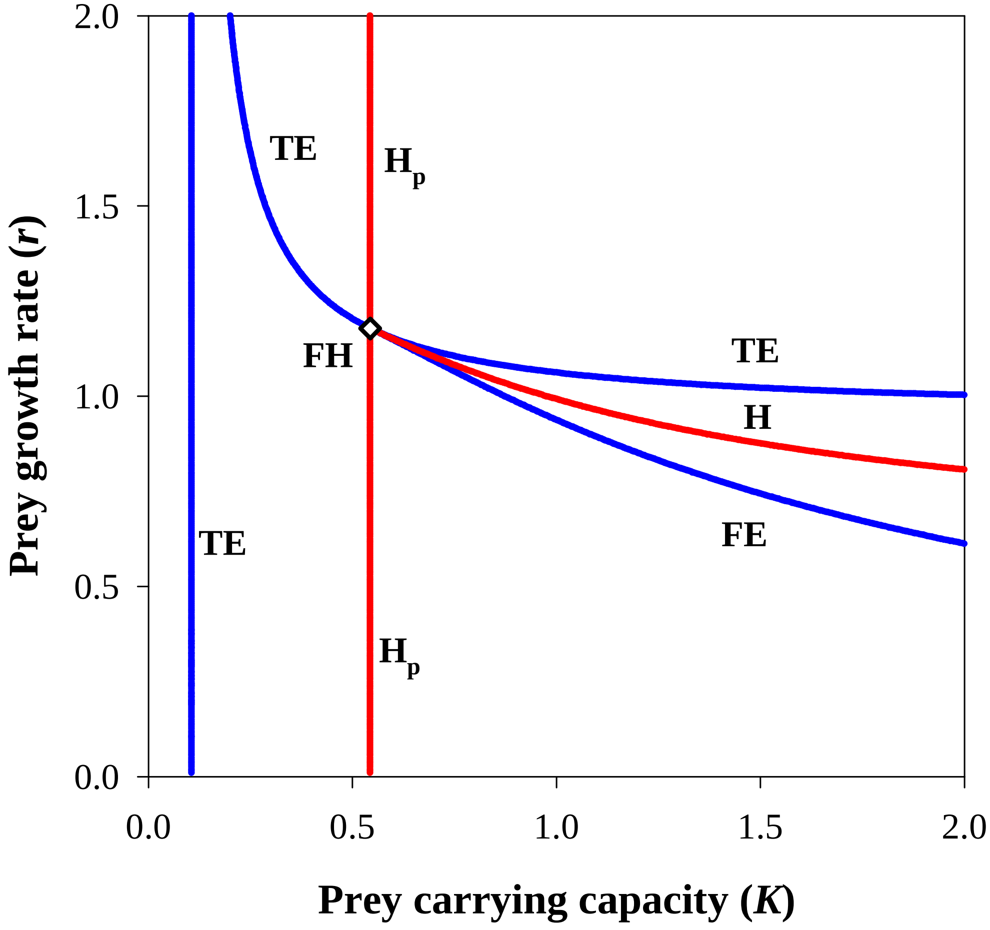
Example Model
K vs. *r* Bifurcation Diagram



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Tri-trophic Model with Sigmoidal Functional Responses

$$\frac{dx_1}{dt} = x_1 r\left(1 - \frac{x_1}{K}\right) - \overbrace{\underbrace{\frac{a_2 x_1^2 x_2}{b_2 + x_1^2}}}$$

Tri-trophic model with a sigmoidal predator and superpredator

$$\frac{dx_2}{dt} = e_2 \overbrace{\underbrace{\frac{a_2 x_1^2 x_2}{b_2 + x_1^2}}} - \frac{a_3 x_2^2 x_3}{b_3 + x_2^2} - d_2 x_2$$

x_1: prey
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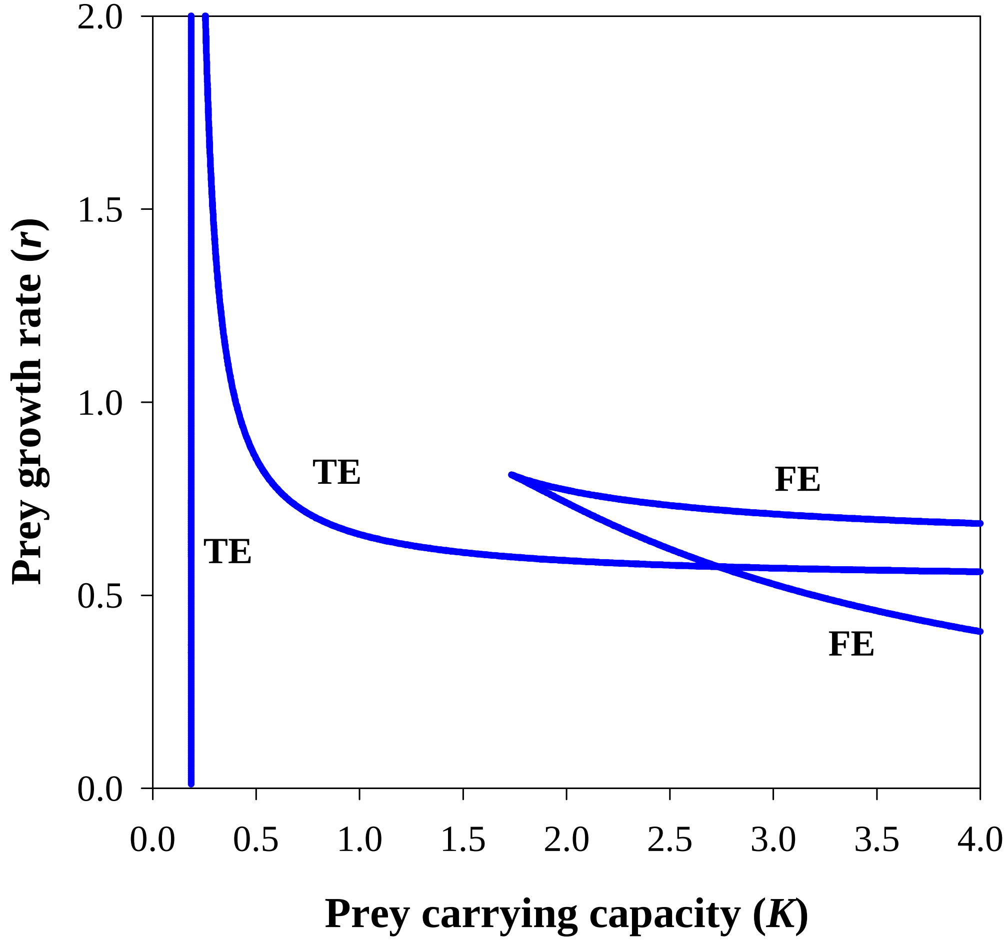
x_3: superpredator

$$\frac{dx_3}{dt} = e_3 \frac{a_3 x_2^2 x_3}{b_3 + x_2^2} - d_3 x_3$$

r : prey growth rate
K : prey carrying capacity

a_i : max predation rate $\quad$ b_i : half saturation constant
d_i : death rate $\quad$ e_i : efficiency

Sigmoidal Model
K vs. *r* Bifurcation Diagram



TE: Transcritical of Equilibria
FE: Fold of Equilibria

Summary

- Interval Newton/Generalized Bisection method is a robust, reliable method to find all solutions to an equation or equation system
- IN/GB can be applied to reliably locate bifurcations of equilibria without *a priori* knowledge of system behavior
  - Initialization independent
  - Capable of locating all branches with mathematical and computational certainty

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Questions?