Trigonometric Functions Review

Circular Representation. Recall that angles of any size can be represented in a unit circle (circle of radius 1). More specifically, we wind the horizontal radius OA (fixed at O) starting from the horizontal axis through the size of the angle we want to depict. For example, to represent the angle $\theta = 3\pi/4$ we rotate the radius OA counter-clockwise till we capture the size of the angle θ. See Figure 1 below. To represent a negative angle rotate the radius OA clockwise. See Figure 2.

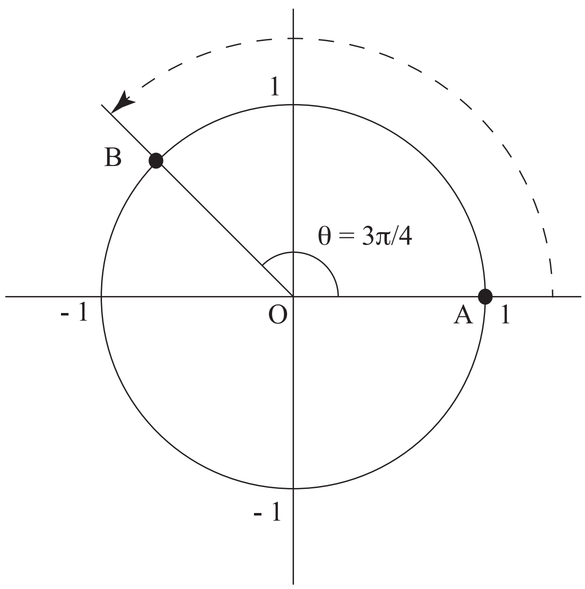


Figure 1.

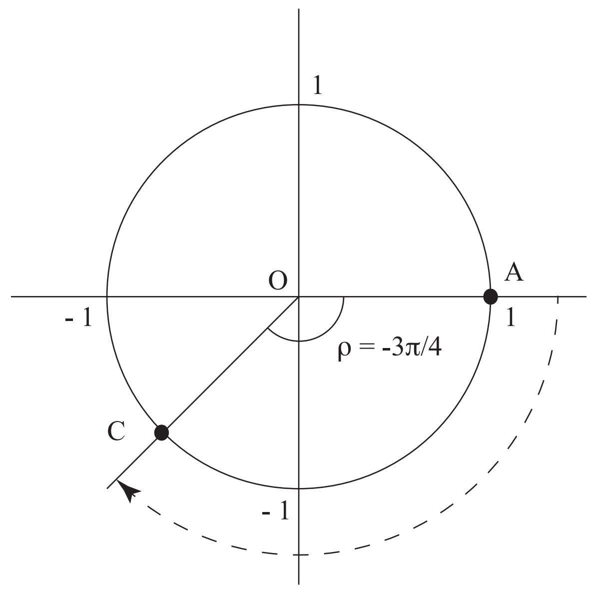
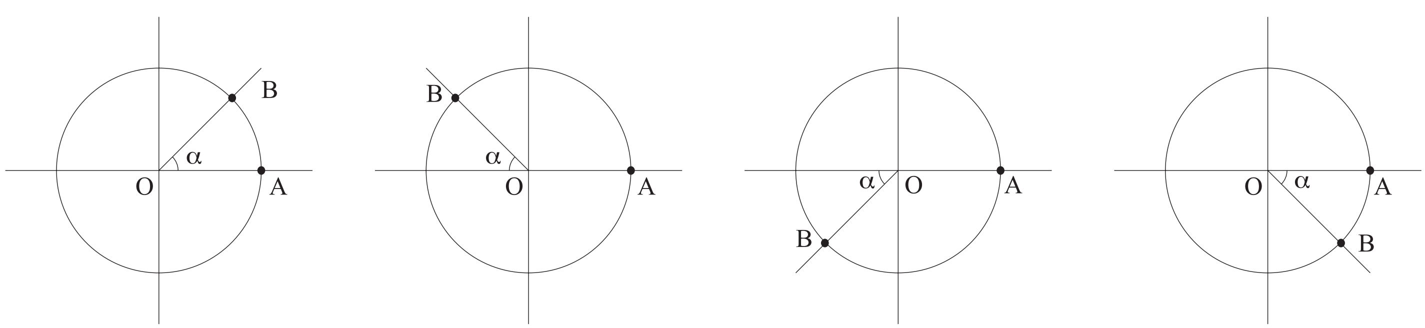


Figure 2.

The point B on the unit circle is associated to the angle $\theta = 3\pi/4$ and the point C on the unit circle is associated to the negative angle $\rho = -3\pi/4$. The radius OB is called the **terminal edge** of the angle $\theta = 3\pi/4$. The terminal edge of the angle $\rho = -3\pi/4$ is OC.

Reference Angle. The reference angle α of a given angle θ is the acute angle ($0 \le \alpha \le \pi/2$) between the terminal edge and the horizontal axis in the circular representation of angle θ. The figures below show the reference angle of an angle whose terminal edge ends in one of the four quadrants.

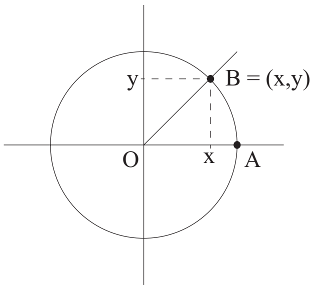


First Quadrant   Second Quadrant   Third Quadrant   Fourth Quadrant

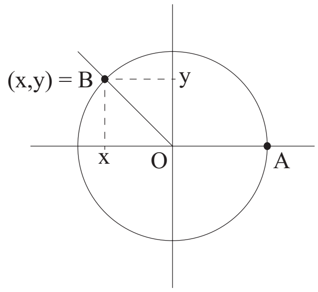
Example 1: The reference angle of $\theta = 3\pi/4$ and $\rho = -3\pi/4$ are both $\pi/4$. Can you see why?

Trigonometric Functions. The trigonometric functions of an angle are given by the coordinates of its associated point on the unit circle as follows:

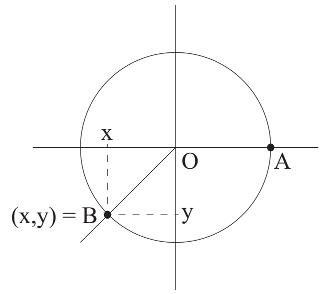
Let the intersection of the terminal edge of angle θ with the unit circle be (x, y). The figure below illustrate the position of (x, y) for angle θ.



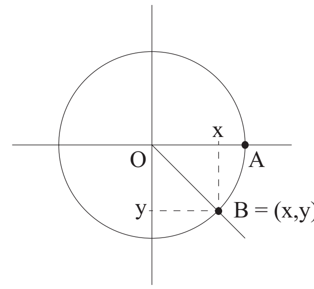
First Quadrant



Second Quadrant



Third Quadrant



Fourth Quadrant

Then the valuation of the sine, cosine and tangent functions at angle θ associated to point (x, y) on the unit circle are given by:

$$\sin\theta = y; \qquad \cos\theta = x; \qquad \tan\theta = \frac{y}{x}$$

Considering the signum ($+$ or $-$) of the coordinates (x, y) in the figures above,

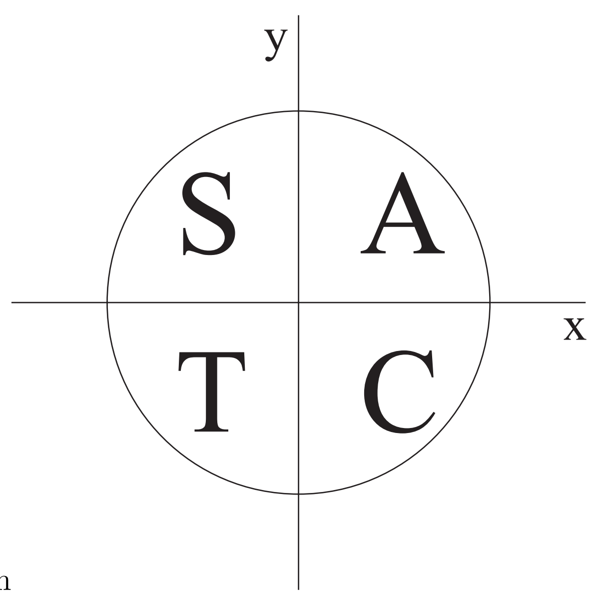
(1) If θ has its terminal edge in the **first** quadrant, then **ALL** trigonometric functions are **positive**. That is $\sin(\theta) > 0$, $\cos(\theta) > 0$, and $\tan(\theta) > 0$.

(2) If θ has its terminal edge in the **second** quadrant, then only the **sine** function is **positive**. That is $\sin(\theta) > 0$, $\cos(\theta) < 0$, and $\tan(\theta) < 0$.

(3) If θ has its terminal edge in the **third** quadrant, then only the **tangent** function is **positive**. That is $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$.

(4) If θ has its terminal edge in the **fourth** quadrant, then only the **cosine** function is **positive**. That is $\sin(\theta) < 0$, $\cos(\theta) > 0$, and $\tan(\theta) < 0$.

You may remember this pattern with the mnemonic illustration below:



CAST Diagram

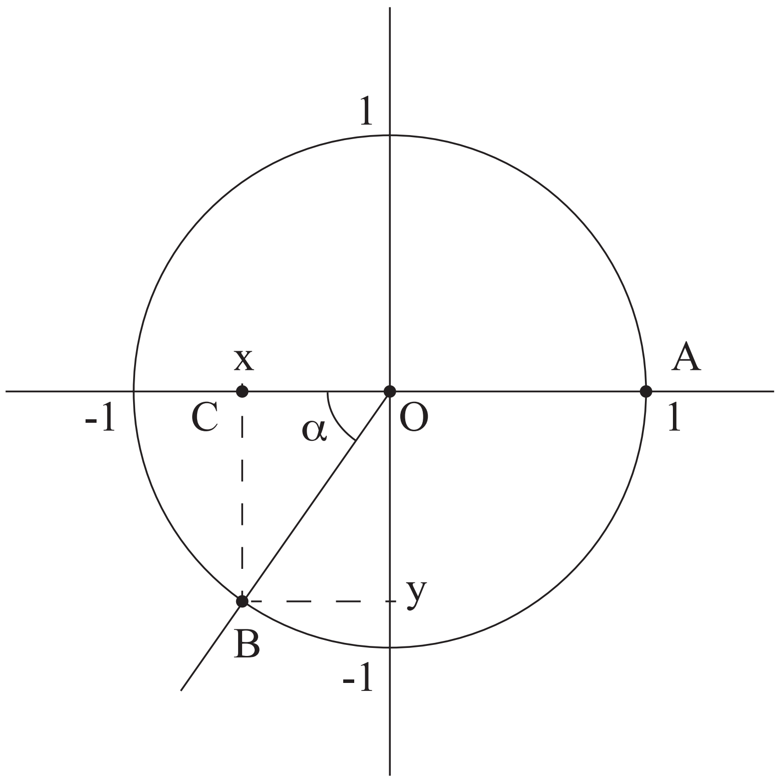
Since the coordinates x and y are between -1 and 1, we see that

$$-1 \leq \sin\theta \leq 1 \qquad \text{and} \qquad -1 \leq \cos\theta \leq 1.$$

Since the $x-$coordinate gets close to 0 and the $y-$coordinate gets close to 1 or -1, near the vertical axis, we see that

$$-\infty < \tan\theta = \frac{y}{x} < \infty.$$

Reference Angle and Trigonometric Functions. The value of a trigonometric function at angle θ can be expressed in terms of the reference angle α of θ. For example, let's look at the case when the terminal edge of θ is in the third quadrant.



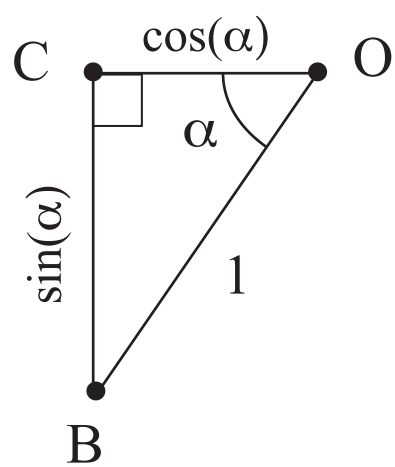
The length of the sides of right-angle triangle OBC is given by

$\overline{OC} = \cos(\alpha)$; $\qquad \overline{BC} = \sin(\alpha)$; $\qquad \overline{OB} = 1$

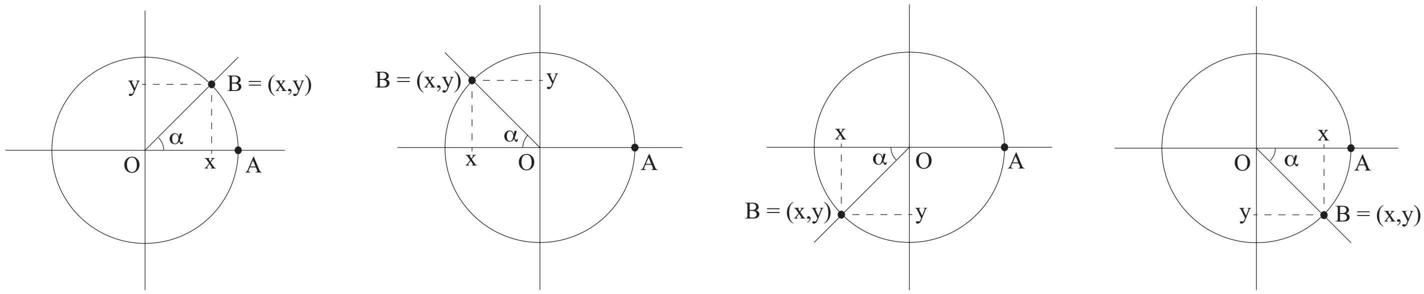
Since the terminal edge of θ is in the third quadrant, both $x-$coordinate and $y-$coordinate are negative. This gives:

$\cos(\theta) = x = -\cos(\alpha)$; $\qquad \sin(\theta) = y = -\sin(\alpha)$

$\tan(\theta) = \dfrac{y}{x} = \tan(\alpha)$



We can carry out the same computation for angles with terminal edges in other quadrants. We summarize our findings below.



First Quadrant Second Quadrant Third Quadrant Fourth Quadrant

Referring to the figures above, if α is the reference angle of θ then the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ can be expressed in terms of α as follows:

(1) θ is in the first quadrant:

$$\sin\theta = \sin\alpha; \qquad \cos\theta = \cos\alpha; \qquad \tan\theta = \tan\alpha$$

(2) θ is in the second quadrant:

$$\sin\theta = \sin\alpha; \qquad \cos\theta = -\cos\alpha; \qquad \tan\theta = -\tan\alpha$$

(3) θ is in the third quadrant:

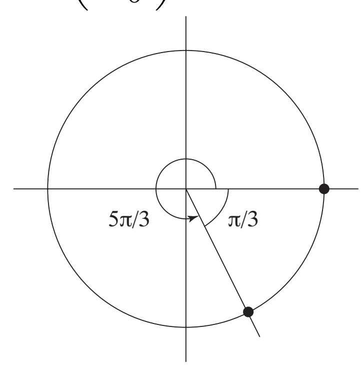
$$\sin\theta = -\sin\alpha; \qquad \cos\theta = -\cos\alpha; \qquad \tan\theta = \tan\alpha$$

(4) θ is in the fourth quadrant:

$$\sin\theta = -\sin\alpha; \qquad \cos\theta = \cos\alpha; \qquad -\tan\theta = \tan\alpha$$

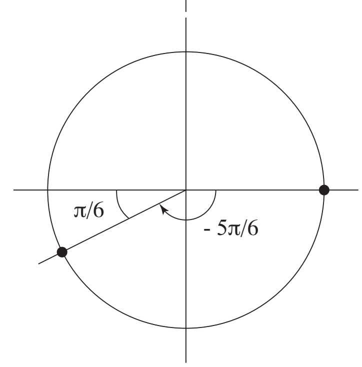
Notice that the signs are according to the CAST diagram above.

Example 2: Find the exact values of (1) $\sin\left(\frac{5\pi}{3}\right)$, (2) $\cos\left(\frac{5\pi}{3}\right)$ and (3) $\tan\left(-\frac{5\pi}{6}\right)$



(1) & (2). $\frac{5\pi}{3}$ is in the fourth quadrant with reference angle $\frac{\pi}{3}$. Sine is negative and cosine is positive in the fourth quadrant. Therefore we have:

$$\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

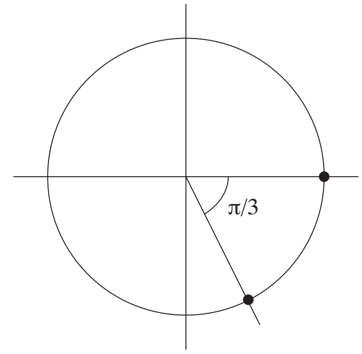


(3). $-\frac{5\pi}{6}$ is in the third quadrant with reference angle $\frac{\pi}{6}$. Tangent is positive in the third quadrant. Therefore we have:

$$\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 3: Find the values of (a) $\arctan\left(-\sqrt{3}\right)$, (b) $\arccos\left(-\dfrac{1}{2}\right)$ and (c) $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$

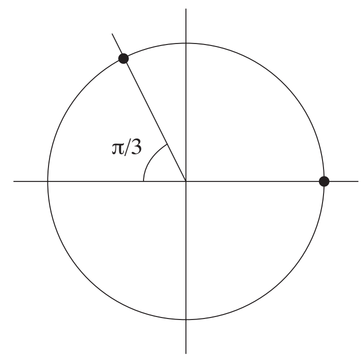
(a) Let $A = \arctan\left(-\sqrt{3}\right)$. Then $\tan(A) = -\sqrt{3}$ and $-\dfrac{\pi}{2} < A < \dfrac{\pi}{2}$.



Since $\tan(A)$ is negative, A must be in the fourth quadrant. The value $\sqrt{3}$ says that the reference angle α of A must be $\dfrac{\pi}{3}$ because $\tan(\alpha) = \sqrt{3}$.

Therefore $A = -\dfrac{\pi}{3}$.

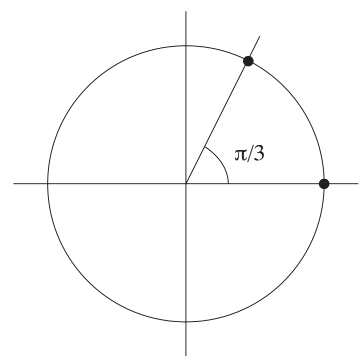
(b) Let $B = \arccos\left(-\dfrac{1}{2}\right)$. Then $\cos(B) = -\dfrac{1}{2}$ and $0 \le B \le \pi$.



Since $\cos(B)$ is negative, B must be in the second quadrant. The value $\dfrac{1}{2}$ says that the reference angle β of B must be $\dfrac{\pi}{3}$ because $\cos(\beta) = \dfrac{1}{2}$.

Therefore $B = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

(c) Let $C = \arcsin\left(\dfrac{\sqrt{3}}{2}\right)$. Then $\sin(C) = \dfrac{\sqrt{3}}{2}$ and $-\dfrac{\pi}{2} \le C \le \dfrac{\pi}{2}$.



Since $\sin(C)$ is positive, C must be in the first quadrant. The only acute angle C such that $\sin(C) = \dfrac{\sqrt{3}}{2}$ is $C = \dfrac{\pi}{3}$.