Math 10350 – Example Set 06A
Section 3.7 The Chain Rule

Definition 1. (The Composite Function) A function $h(x)$ is said to be a composite function of $g(x)$ followed by $f(x)$ if $h(x) = f(g(x))$. We may write: $h : x \overset{g}{\longmapsto}$ ________ $\overset{f}{\longmapsto}$ ________

1. Find functions $f(x)$ and $g(x)$, not equal x, such that $h(x) = f(g(x))$:

(a) $h(x) = (x^4 + 2x^2 + 7)^{21}$ $\qquad h : x \overset{g}{\longmapsto}$ ____________ $\overset{f}{\longmapsto}$ ____________

Ans: $f(x) \overset{?}{=}$ ____________ and $g(x) \overset{?}{=}$ ____________

(b) $h(x) = \sin(x^2 + 1)$ $\qquad h : x \longmapsto$ ____________ $\longmapsto$ ____________

Ans: $f(x) \overset{?}{=}$ ____________ and $g(x) \overset{?}{=}$ ____________

World Guinness Record Approved Text: "The razor-toothed piranhas of the genera Serrasalmus and Pygocentrus are the most ferocious freshwater fish in the world. In reality they seldom attack a human."

Think about it: In a competition for the title of "Fastest Text Messager", it is observed that Competitor A inputs text three times faster than B, and Competitor B inputs text two times faster than C. How much faster is Competitor A than Competitor C? Why?

A – – $\underset{\text{Rate of } A \text{ relative to } B}{3}$ – – B – – $\underset{\text{Rate of } B \text{ relative to } C}{2}$ – – C

– – – – – – – – – ? Rate of A relative to C ? – – – – – – – – –

The Chain Rule. Suppose $y = f(g(x))$. To find a formula for $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))]$, we set $u = g(x)$ then $y = f(u)$.

y – – $\underset{\text{Rate of } y \text{ relative to } u}{\frac{dy}{du}}$ – – u – – $\underset{\text{Rate of } u \text{ relative to } x}{\frac{du}{dx}}$ – – x

– – – – – – – – – $\underset{\text{? Rate of } y \text{ relative to } x \text{ ?}}{\frac{dy}{dx} \overset{?}{=}}$ – – – – – – – – –

So we expect $\boxed{\dfrac{dy}{dx} = \qquad\qquad}$

Our guess is in fact correct, and the formula for $\dfrac{dy}{dx}$ is called the **Chain Rule** (in Leibniz notation).

But $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))] = [f(g(x))]'$, $\dfrac{dy}{du} = f'(u) = f'(g(x))$ and $\dfrac{du}{dx} = g'(x)$. Thus we also have:

$$\boxed{\frac{d}{dx}[f(g(x))] = [f(g(x))]' = \qquad\qquad\qquad\qquad}$$

1. Find the coordinates of the points on the curve $y = 2xe^{-x^2}$ where the tangent lines are horizontal.

2. Find the derivative of the following functions:

a. $f(x) = \sqrt[3]{1 + x^3}$

b. $g(x) = \cot 5x$

c. $h(x) = \dfrac{1}{2}x^2\sqrt{16 - x^2}$

d. $R = \csc^3(\pi x)$

e. $w = \left(\dfrac{t+1}{t^2+2}\right)^4$

f. $y = \tan^3(2x^2 + 1)$

Math 10350 – Example Set 06B
Section 3.7 The Chain Rule
Section 3.9 Derivative of the Natural Log

1. Consider the functions $f(x) = e^x$ and $g(x) = \ln x$.

a. Sketch the graph of $f(x) = e^x$ and $g(x) = \ln x$ on the same set of axes. What could you say about their relationship? How are $f(x)$ and $g(x)$ related?

b. Using the fact that $\frac{d}{dx}(e^x) = e^x$ and the chain rule, find a formula for $\frac{d}{dx}(\ln x)$.

c. Using the change of base formula $\log_b x = \frac{\ln x}{\ln b}$, show that $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

2. Find the equation of the tangent line to the graph of $y = \frac{\ln x - 1}{\ln x + 1}$ when $x = 1$.

3. Find the derivatives of the following functions:

a. $f(\theta) = \ln(\sin\theta + 2)$

b. $y = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$

c. $g(z) = \ln(\ln z)$ for $z > 1$.

d. $y = e^{(\ln x)^3}$

e. $x^e + e^x$

e. x^x

Math 10350 – Example Set 06C
Section 3.8 Implicit Differentiation
including Logarithmic Differentiation

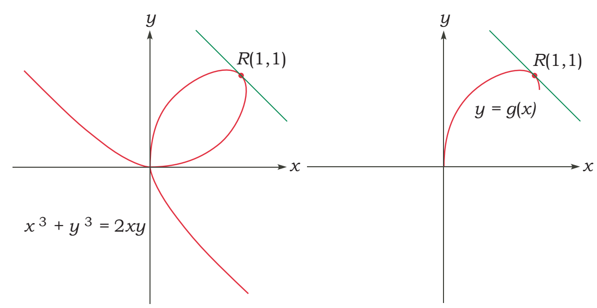
1. Find the derivative of the given functions:

(a) $(2x+1)^{\cos(e)}$ (b) $(2e+1)^{\cos x}$ (a) $(2x+1)^{\cos x}$

2. Find the equation of the tangent line at the point $P(1,2)$ on the circle $x^2+y^2=5$ by solving for y as an appropriate expression of x.

Remark: For a general relation between x and y, it is difficult to write y as a function of x. For example, $x^3+y^3=2xy$. To find the slope at $R(1,1)$ on the curve using the above method, we need to find **explicitly** $g(x)$. This is very hard!!

We say that y is an implicit function of x. To find $\dfrac{dy}{dx}$ in such situation we employ a powerful method called **Implicit Differentiation**.



3. Verify that the point $(1,1)$ is on the curve $x^3+y^3=2xy$. Find (a) $\dfrac{dy}{dx}$, (b) the slope of the curve at $(1,1)$, and (c) the point(s) on the curve where the tangent line is horizontal.

(b) $y=-x+2$; (c) $\left(\dfrac{16^{1/3}}{3}, \dfrac{16^{2/3}}{6}\right)$

4. Find $\dfrac{dy}{dx}$ if $\cos(xy)=x+y^2$.

Math 10350 – Example Set 06C
Power Functions, Exponential Functions, and Mixing them.

1. Determine whether the following functions are of the form $[f(x)]^n$, $a^{g(x)}$, and $[f(x)]^{g(x)}$ where a and n are constants, and $f(x)$ and $g(x)$ are functions of x. Find their derivatives.

a. $y = (2x^2 + 5)^{e^2+3}$

b. $y = (2x^2 + 5)^{e^x+3}$

c. $y = (2\pi^2 + 5)^{x^2+3}$

d. $y = (\sin(e) + \cos^2(e))^{x^2}$

e. $y = (\sin(e) + \cos^2(e))^{\pi^2}$

f. $y = (\sin(x) + \cos^2(e))^{e^2}$