10350 Derivative Check Name ____________________

1. Find the first and second derivatives of the following functions. Simplify each of your answers as far as possible.

(a) $f(x) = x - 5(x-2)^{1/5}$

$$f'(x) \stackrel{?}{=} 1 - 5 \cdot \frac{1}{5}(x-2)^{-4/5} \cdot 1$$

$$= -(x-2)^{-4/5}$$

$$f''(x) \stackrel{?}{=} -\left(-\frac{4}{5}\right)(x-2)^{-9/5} \cdot 1$$

$$= \frac{4}{5}(x-2)^{-9/5}$$

(b) $g(x) = xe^{-x^2}$

$$g'(x) \stackrel{?}{=} x \cdot e^{-x^2} \cdot (-2x) + 1 \cdot e^{-x^2}$$

$$= -2x^2 e^{-x^2} + e^{-x^2}$$

$$= (-2x^2 + 1) \cdot e^{-x^2}$$

$$g''(x) \stackrel{?}{=} (-2x^2 + 1)e^{-x^2} \cdot (-2x) + (-4x)e^{-x^2}$$

$$= (4x^3 - 2x - 4x)e^{-x^2}$$

$$= (4x^3 - 6x)e^{-x^2}$$

(c) $y = \dfrac{e^{3x} - 1}{e^{3x} + 1}$

$$\frac{dy}{dx} \stackrel{?}{=} \frac{(e^{3x}+1)(3e^{3x}) - (e^{3x}-1)(3e^{3x})}{(e^{3x}+1)^2}$$

$$= \frac{3e^{3x}\left(e^{3x}+1-(e^{3x}-1)\right)}{(e^{3x}+1)^2}$$

$$= \frac{3e^{3x}\left(e^{3x}+1-e^{3x}+1\right)}{(e^{3x}+1)^2}$$

$$= \frac{3e^{3x}(2)}{(e^{3x}+1)^2} = \frac{6e^{3x}}{(e^{3x}+1)^2}$$

$$\frac{d^2y}{dx^2} \stackrel{?}{=} \frac{(e^{3x}+1)^2(18e^{3x}) - 2(e^{3x}+1)(3e^{3x})(6e^{3x})}{(e^{3x}+1)^4}$$

$$= \frac{18e^{3x}(e^{3x}+1)\left[e^{3x}+1-2e^{3x}\right]}{(e^{3x}+1)^4}$$

$$= \frac{18e^{3x}(1-e^{3x})}{(e^{3x}+1)^3}$$

Note

10350 Exam 03 Review Summary Name ______________________

1. The statement: "$f(x)$ is increasing on $a < x < b$." then

1a. "$f'(x)$ is Positive on $a < x < b$."

2. The statement: "$f'(x)$ is negative on $a < x < b$." then

2a. "$f(x)$ is decreasing on $a < x < b$."

2b. "The slope of the graph of $f(x)$ is negative on $a < x < b$."

3. The statement: "The graph of $f(x)$ is concave up on $a < x < b$." is the same as:

3a. "$f''(x)$ is positive on $a < x < b$." is the same as:

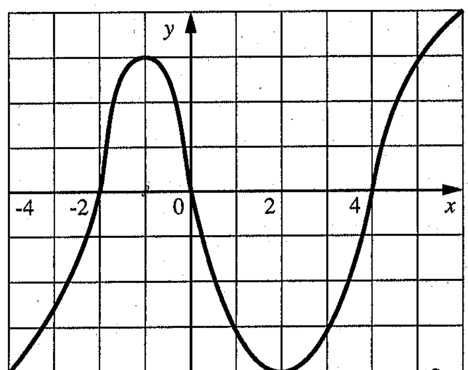
3b. "$f'(x)$ is increasing on $a < x < b$."

4. The statement: "$f'(x)$ is decreasing on $a < x < b$." is the same as:

4a. "$f''(x)$ is negative on $a < x < b$." is the same as:

4b. "The graph of $f(x)$ is concave down on $a < x < b$."

5. The figure below is the graph of the **derivative** $f'(x)$ of $f(x)$ for $-4 < x < 6$. Find all intervals on which **the graph of** $f(x)$ is concave up?



(i) Find all values of x in $(-4, 6)$ for which $f(x)$ is increasing. f increasing $\Rightarrow f' > 0$

$-2 < x < 0$ or $4 < x < 6$

(ii) Find the critical points of $f(x)$ in $(-4, 6)$. Are these local maximums or minimums?

$x = -2$ (local min) $x = 0$ (local max) $x = 4$ (local min)

$f' < 0$ | $f' > 0$ sign of f' — 2

$f' > 0$ | $f' < 0$ — 0

$f' < 0$ | $f' > 0$ — 4

(iii) Find all intervals on which **the graph of** $f(x)$ is concave up in $(-4, 6)$.

$f'' > 0 \Leftrightarrow f'$ increasing

$-4 < x < -1$ or $2 < x < 6$

(iv) Find all values of x in $(-4, 6)$ for which $f(x)$ has an inflection point.

$x = -1, 2$ ← places such that $f''(x)$ changes sign.

Definition: Let $f(x)$ be defined at c that is $f(c)$ is a real number.

We say that c is a **critical point** of $f(x)$ if (1) $f'(c) = 0$ or (2) $f'(c)$ does not exist.

We say that c is a **inflection point** of $f(x)$ if the graph of $f(x)$ changes concavity at $x=c$

The extreme value theorem

If $f(x)$ is continuous on a closed and bounded interval $a \le x \le b$ then $f(x)$ attain global minimum and global maximum on for some values of x in $a \le x \le b$.

On a closed and bound interval $a \le x \le b$, a continuous function $f(x)$ attains its absolute maximum and absolute minimum occur at

(1) Critical points of $f(x)$, or (2) x where $f(x)$ is undefined.

The First Derivative Test

Suppose $f(x)$ has a critical point at $x = c$. We classify the critical point as follows:

- if $f'(x)$ changes its sign from positive to negative at $x = c$, then there is a relative (local) maximum at $x = c$. / \
- if $f'(x)$ changes its sign from negative to positive at $x = c$, then there is a relative (local) minimum at $x = c$. \ /
- if $f'(x)$ does not change its sign on both sides of $x = c$, then there is neither a relative (local) minimum nor a relative (local) maximum at $x = c$.

Second Derivative Test

Let $f(x)$ be a smooth function such that $f'(c) = 0$.

- If $f''(c) > 0$ then f has a local minimum at the point $(c, f(c))$.
- If $f''(c) < 0$ then f has a local maximum at the point $(c, f(c))$.
- If $f''(c) = 0$ then the test fails. Use first derivative test.

Math 10350 – Monotonicity Example

The only possible values of x at which the monotonicity of a function $f(x)$ changes are:

(1) critical points and (2) $f(x)$ is undefined.

6. Find all values of x for which $f(x) = x - 5(x-2)^{1/5}$ is increasing or decreasing with the steps outlined below. Classify all critical points using first derivative test.

Step 1: Find all **critical points** of f. (That is all points c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.)

$$f'(x) = 1 - (x-2)^{-4/5} = 1 - \frac{1}{(x-2)^{4/5}}$$

$f'(x)$ is undefined at $x = 2$ but $f(2) = 2 - 5(0)^{1/5} = 2 \Rightarrow x = 2$ is a critical point.

$$f'(x) = 0 \Rightarrow 1 - \frac{1}{(x-2)^{4/5}} \Rightarrow \frac{1}{(x-2)^{4/5}} = 1 \Rightarrow (x-2)^{4/5} = 1$$

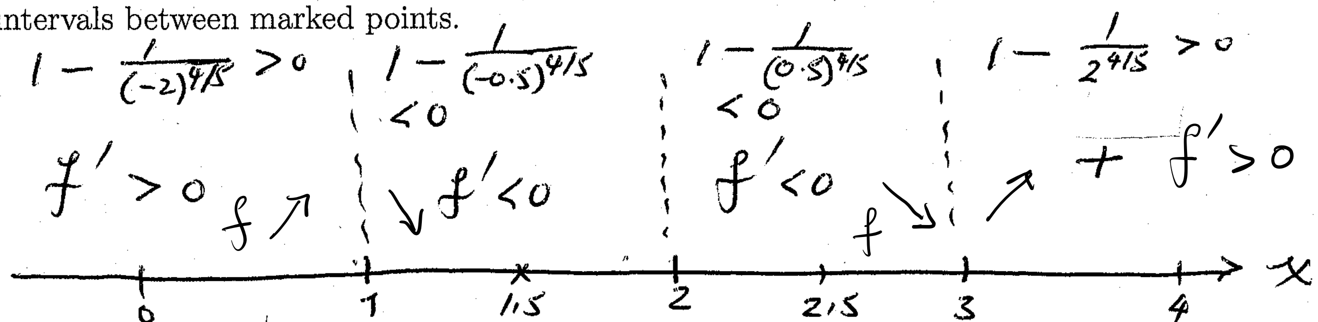
$$\Rightarrow [(x-2)^{4/5}]^5 = 1^5 \Rightarrow (x-2)^4 = 1 \Rightarrow x - 2 = \pm\sqrt[4]{1}$$

$$\Rightarrow x - 2 = -1 \text{ or } -1 \Rightarrow x = 2-1 \text{ or } 2+1 = 1 \text{ or } 3$$

So $x = 1, 2, 3$ are critical points

Step 2: Find points where f have a **vertical asymptote** or undefined. Answer: NONE

Step 3: Draw a number line, mark all points found in Steps 1 and 2, and find the sign of $f'(x)$ in each intervals between marked points.



Step 4: Write down the values of x for which f is increasing ($f'(x) > 0$) and those for which f is decreasing ($f'(x) < 0$).

$f(x)$ is decreasing on $(1, 2) \cup (2, 3)$

$f(x)$ is increasing on $(-\infty, 1) \cup (3, \infty)$

Step 5: Classify all critical points using first derivative test.

$x = 1$ (local maximum), $x = 2$ (neither local maximum nor minimum)

$x = 3$ (local minimum)

The only possible values of x at which the concavity of a function $f(x)$ changes are:

(1) $f''(x) = 0$, (2) $f''(x)$ does not exist. and (3) $f(x)$ does not exist .

7. Find all values of x for which $g(x) = xe^{-x^2}$ is increasing or decreasing with the steps outlined below. Classify all critical points using first derivative test.

Step 1: Find all ~~all~~ points c in the domain where $g''(c) = 0$ or $g''(c)$ does not exist. $\left(g''(x) = (4x^3 - 6x)e^{-x^2}\right)$

$g''(x) = (4x^3 - 6x)e^{-x^2} = 0 \Rightarrow 4x^3 - 6x = 0$

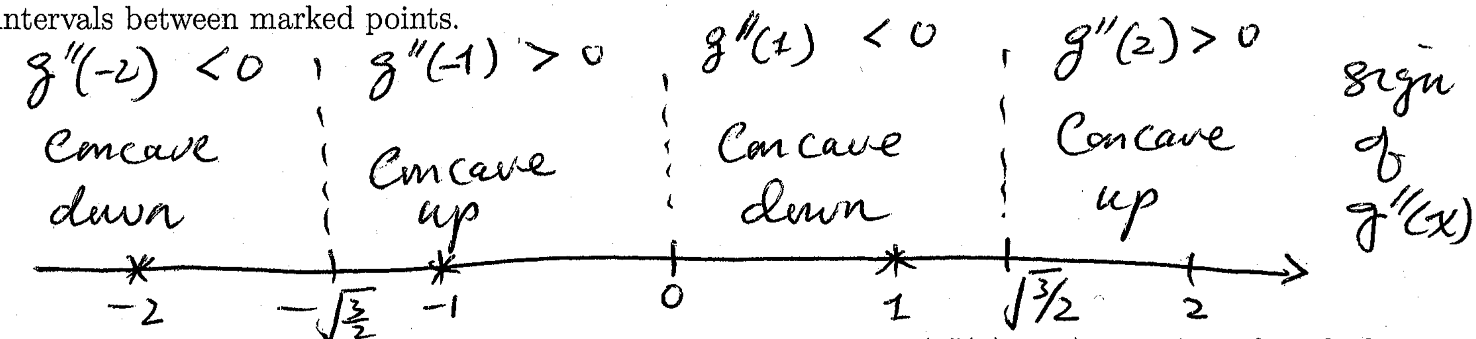
$\Rightarrow 2x(2x^2 - 3) = 0 \Rightarrow x = 0$ or $x^2 = \frac{3}{2} \Rightarrow x = \pm\sqrt{\frac{3}{2}}$

$\Rightarrow x = 0, -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}$.

$g''(x)$ exists for all x.

Step 2: Find points where g have a **vertical asymptote** or undefined. Answer: None

Step 3: Draw a number line, mark all points found in Steps 1 and 2, and find the sign of $g''(x)$ in each intervals between marked points.



Step 4: Write down the values of x for which g is concave up ($g''(x) > 0$) and those for which g is concave down ($g''(x) < 0$).

Concave down : $\left(-\infty, -\sqrt{\frac{3}{2}}\right) \cup \left(0, \sqrt{\frac{3}{2}}\right)$

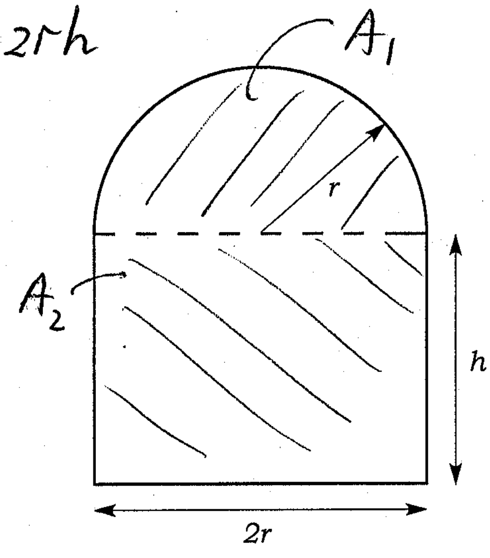
Concave up : $\left(-\sqrt{\frac{3}{2}}, 0\right) \cup \left(\sqrt{\frac{3}{2}}, \infty\right)$

Step 5: Find all inflection points for the function $g(x)$.

$x = -\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}$ since $g''(x)$ changes sign at these points.

8. A Norman window has a semi-circular portion mounted (exactly) on one side of a rectangle as show below. Answer the following questions if the perimeter of the window is 50 ft and r is the radius of the circular portion. Find the dimensions of the window that lets the (i) least light in and (ii) most light in.

Area of the window, $A = A_1 + A_2 = \dfrac{\pi r^2}{2} + 2rh$



Given perimeter : $50 = \dfrac{2\pi r}{2} + h + 2r + h$

so $50 = \pi r + 2r + 2h$

Optimize A under the constraint $50 = \pi r + 2r + 2h$

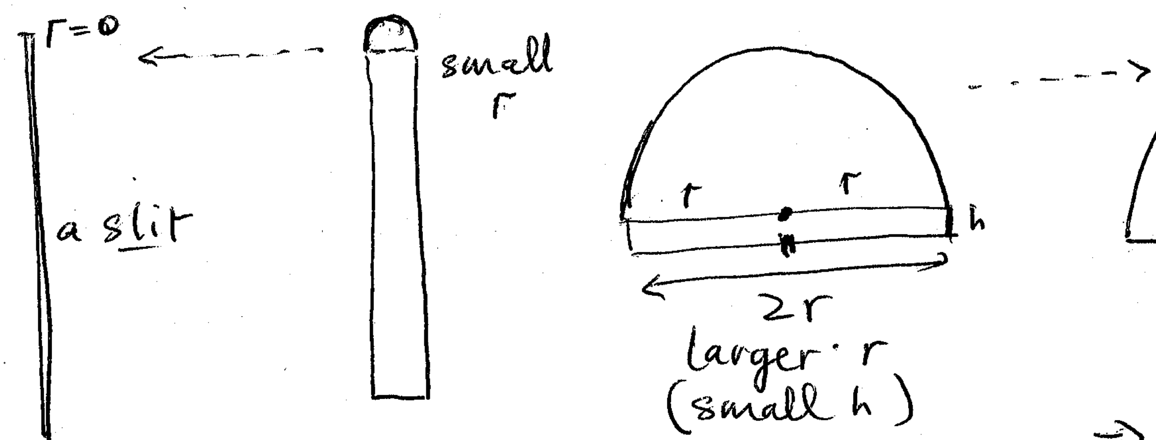
$50 = \pi r + 2r + 2h \Rightarrow 2h = 50 - \pi r - 2r$

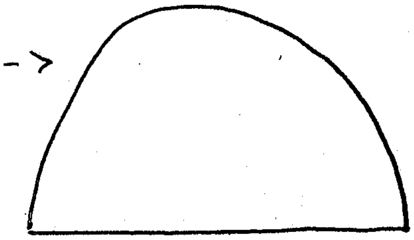
$\Rightarrow h = \dfrac{50 - \pi r - 2r}{2}$

$$A(r) = \frac{\pi r^2}{2} + 2rh = \frac{\pi r^2}{2} + 2r\left(\frac{50 - \pi r - 2r}{2}\right)$$

$$= \frac{1}{2}\pi r^2 + r(50 - \pi r - 2r) = \frac{1}{2}\pi r^2 + 50r - \pi r^2 - 2r^2$$

$$= -\frac{1}{2}\pi r^2 - 2r^2 + 50r$$





At largest r

$h = 0$

$\Rightarrow 50 = \pi r + 2r$

$= (\pi + 2) r$

$\Rightarrow r = \dfrac{50}{\pi + 2}$

Optimize $A(r)$ over the interval $0 \leq r \leq \dfrac{50}{\pi + 2}$

<u>Check End point:</u> $A(0) = 0$

$$A\left(\frac{50}{\pi+2}\right) = -\frac{1}{2}\pi\left(\frac{50}{\pi+2}\right)^2 - 2\left(\frac{50}{\pi+2}\right)^2 + 50\left(\frac{50}{\pi+2}\right)$$

$$= \frac{50}{\pi+2}\left[-\frac{\pi}{2}\cdot\frac{50}{\pi+2} - 2\cdot\frac{50}{\pi+2} + 50\right]$$

$$= \frac{50}{\pi+2}\left[\frac{-25\pi}{\pi+2} - \frac{100}{\pi+2} + \frac{50(\pi+2)}{\pi+2}\right]$$

$$= \frac{50}{\pi+2}\left[\frac{-25\pi - 100 + 50\pi + 100}{\pi+2}\right] = \frac{50}{\pi+2}\cdot\frac{25\pi}{\pi+2}$$

$$= \frac{1250\pi}{(\pi+2)^2}$$

<u>Check Critical point:</u> $A'(r) = -\pi r - 4r + 50$

$$A'(r) = 0 = -(\pi+4)r + 50 \Rightarrow r = \frac{50}{\pi+4}$$

$$A\left(\frac{50}{\pi+4}\right) = -\frac{1}{2}\pi\left(\frac{50}{\pi+4}\right)^2 - 2\left(\frac{50}{\pi+4}\right)^2 + 50\left(\frac{50}{\pi+4}\right)$$

$$= \frac{50}{\pi+4}\left[-\frac{\pi}{2}\cdot\frac{50}{\pi+4} - 2\cdot\frac{50}{\pi+4} + 50\right]$$

$$= \frac{50}{\pi+4}\left[\frac{-25\pi - 100}{\pi+4} + 50\right] = \frac{50}{\pi+4}\left[\underbrace{\frac{-25(\cancel{\pi+4})}{\cancel{\pi+4}} + 50}_{25}\right]$$

$$= \frac{1250\pi}{\pi+4}$$

We check $\quad A(0) < A\left(\frac{50}{\pi+2}\right) < A\left(\frac{50}{\pi+4}\right)$

At max A, $\quad r = \frac{50}{\pi+4}, \quad h = \frac{50 - (\pi+2)\left(\frac{50}{\pi+4}\right)}{2} = \frac{50}{\pi+4}$

$$= \frac{50\left(1 - \frac{\pi+2}{\pi+4}\right)}{2}$$

6