**Math 10350 – Exam 02 Review**

**1.** A differentiable function $g(t)$ is such that $g(2) = 2, \quad g'(2) = -1, \quad g''(2) = 1/2.$

(a) If $p(t) = g(t)e^{t^2}$ find $p'(2)$ and $p''(2)$. (Ans: $p'(2) = 7e^4$; $p''(2) = 28.5e^4$)

(b) If $f(t) = \sec(2\pi t) \cdot g(t)$ find the slope of the tangent line to the graph of $f(t)$ at $t = 2$. (Ans: $f'(2) = -1$)

(c) If $h(t) = \dfrac{1 - t^2 g(t)}{t^2}$ find the derivative of $h(t)$ at $t = 2$. Hint: No quotient rule needed. (Ans: $h'(2) = 3/4$)

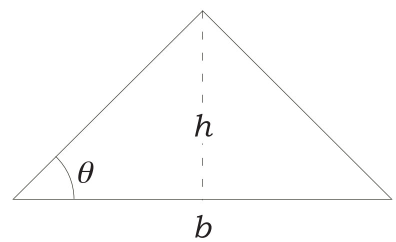
(d) Let $s(t) = \cos(\pi g(t))$ be the position of a particle moving on a straight line, find its velocity and acceleration at the moment when $t = 2$. (Ans: $s'(2) = 0$; $s''(2) = -\pi^2$)

(e) If $q(t) = \ln\left(\dfrac{4 - g(t)}{4 + g(t)}\right)$ find the instantaneous rate of change of $q(t)$ at $t = 2$. Hint: No quotient rule needed. (Ans: $q'(2) = 2/3$)

(f) Find the linearization of $g(t)$ at $t = 2$ then estimate the value of $g(1.8)$? (Ans: $g(1.8) = 2.2$)

**2.** Consider a cylindrical metal rod is heated up in a furnace. When the volume of the rod is $80\pi$ cm$^3$ and its height is 5 cm, both radius and height are growing at a rate of 0.5 cm/min. At what rate is the volume is growing? (Ans: At required moment $r = 4$ cm and so $dV/dt = 28\pi$ cm$^3$/min.)

**3.** An **isosceles** triangle (see figure below) with **fixed** area of 50 cm$^2$ has its height decreasing at a rate of 0.1 cm/sec. At what rate is the base of the triangle changing at the instant when its height is 5 cm? How fast is the angle $\theta$ changing at the same instant? (Hint: You do not need to find $\theta$ explicitly.)




(Ans: $db/dt = 0.4$ cm/sec. Note that $\tan\theta = h/(b/2) = h^2/50$. $d\theta/dt = -0.016$ radian/sec.)

**4.** Two cars start from the same intersection at the same time. Car A heads east at a constant speed of 40 miles per hour, and Car B heads north at a speed of 30 miles per hour. (a) Find the distance $s$ between Car A and Car B in terms of time $t$. (b) How fast is the distance between the cars changing? (c) If car B is speeding according to the position $x(t) = 10t^2$ miles (measured from the intersection), how would your answers in (a) and (b) change? (Ans: (a) $s(t) = 50t$; (b) $s'(t) = 50$; (c) $s(t) = 10(16t^2 + t^4)^{1/2}$; $s'(t) = 5(16t^2 + t^4)^{-1/2}(32t + 4t^3)$)

**5.** If $y - 4\cos(y) + 3xy^2 = x^2 - 8$ find $\dfrac{dy}{dx}$ and the tangent line to the curve at the point $(2, 0)$.

$\left(\text{Ans: } y' = \dfrac{2x - 3y^2}{4\sin(y) + 6xy + 1};\ y = 4x - 8\right)$

**6.** A block of ice with a square base has dimension $x$ inches by $x$ inches by $3x$ inches.

(a) If the block of ice is melting so that its surface area $A$ is decreasing at a rate of 2 in$^2$/sec, find the rate at which $x$ is changing when $x = 12$ inches. (Ans: $dx/dt = -1/168$ in/sec)

(b) Estimate the change in volume when $x$ changes from 12 in to 12.5 in. What is the corresponding percentage change in volume. (Ans: $\Delta V = V'(12)\Delta x = 648$ in$^3$; $\frac{\Delta V}{V} \times 100\% = 12.5\%$.)

**7.** A differentiable function $g(x)$ is such that

$$g(2) = -1, \qquad g'(2) = 2, \qquad f(2) = 2 \quad \text{and} \quad f'(2) = -4$$

**(a)** If $A(x) = 2g(x) + 3f(x) + e^2$ find $A'(2) \stackrel{?}{=}$ ____________.

**(b)** If $B(x) = g(x) \cdot (2e^x - 3)$ find the slope of the graph of $B(x)$ at $x = 2$.

**(c)** If $C(x) = \dfrac{4x + g(x)}{f(x) + g(x)}$ find the instantaneous rate of change of $C(x)$ at $x = 2$

**8.** Find the third derivative of the following functions: (a) $y = 3\tan(3\theta)$; (b) $y = (1 - t)^{5/2}$; (c) $y = x \cdot 3^x$

(Ans: (a) $324\sec^2(3\theta)\tan^2(3\theta) + 162\sec^4(3\theta)$; (b) $-\frac{15}{8}(1 - t)^{-1/2}$; (c) $(\ln 3)^2(x \cdot 3^x \ln 3 + 3^{x+1})$)

**9.** A point is moving on the curve $x^3 + y^3 = xy + 1$. If at the point $(1, -1)$, the velocity of the point in the $x-$direction is $-2$ units per minute, what is its velocity in the $y$-direction? (Ans: $dy/dt = 4$ unit/min)

**10.** Using limits find the derivative of $f(x) = \sqrt{x+1}$. Write down the linear approximation to $f(x)$ at $x = 3$. Estimate $\sqrt{3.8}$. Draw a graph to illustrate your estimation. (Ans: $f(x) \approx \frac{1}{4}(x-3) + 2$ for $x$ near 3; $\sqrt{3.8} \approx 1.95$)

**11.** Find the derivatives of the following functions: (a) $y = (1 + x^2)^x$; (b) $y = e^{e^x}$; (c) $y = x^{x^2}$.
(Ans: (a) $(1 + x^2)^x \left[\frac{2x^2}{1+x^2} + \ln(1 + x^2)\right]$; (b) $e^{e^x + x}$; (c) $x^{x^2}(x + 2x \ln x)$)

**12.** Find the value of $k$ such that the following function $f(x)$ is continuous at $x = 0$:

$$f(x) = \begin{cases} \dfrac{\sin(9x^2)}{3x} & x \neq 0 \\ k & x = 0 \end{cases}$$

Show your work using limits very carefully. (Ans: $k = 0$)

**13.** Find the values of the following limits:

(a) $\displaystyle\lim_{x\to 0^+} x \cos \frac{1}{x^2}$; (b) $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{1-x}$; (c) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}$; (d) $\displaystyle\lim_{x\to 0} \frac{\sin^2 5x}{\sin 3x}$. (Ans: (a) 0; (b) $-1$; (c) 5/3; (d) 0)

**14.** Find $\dfrac{dy}{dx}$ if $e^{xy} + y^2 + x = 1$. $\left(\text{Ans: } \dfrac{dy}{dx} = \dfrac{-ye^{xy} - 1}{xe^{xy} + 2y}\right)$

**15.** At noon ship $A$ is 8 km west from ship $B$. Ship A is sailing south at 5 km per hour and ship $B$ is sailing north at 3km per hour.
(a) Find a formula for the distance $L$ between the two ships. (Ans: $L(t) = 8\sqrt{t^2 + 1}$ km)
(b) How fast is the distance between the ships changing at 1p.m.? (Ans: $4\sqrt{2}$ km/h)

**16.** Consider the curve given by the parametric equations:

$$x = t \sin t; \qquad y = t \cos t.$$

(a) How fast are the coordinates $x$ and $y$ changing (relative to $t$) when $t = \pi$? (b) Find the equation of the tangent line at the point when $t = \pi$. Give your answer in the form $y = mx + b$.
$\left(\text{Ans: (a)} x'(\pi) = -\pi, y'(\pi) = -1; \text{(b)} y = \dfrac{x}{\pi} - \pi\right)$

**17.** Find $\dfrac{dy}{dx}$ for the given parametric equations below. Using your result, find (a) the equation of the tangent line to the curve at the given $t = t_0$, and (b) the values of $t$corresponding to the points on curve where the tangent lines are horizontal.

**17 (i).** $x = e^{t^3} + t$; $\quad y = e^{2t} - 4t + 1$; $\quad t = 0$. (Ans: (a) $y = -2x + 4$; (b) $t = (\ln 2)/2$)

**17 (ii).** $x = \sec\left(\dfrac{t}{2}\right)$; $\quad y = \cos(t) + \dfrac{t}{2}$; $\quad t = \dfrac{\pi}{2}$. Here restrict $0 < t < 2\pi$.
(Ans: (a) $y = -\frac{x}{\sqrt{2}} + 1 + \frac{\pi}{4}$; (b) $t = \pi/6, 5\pi/6$)

**Math 10350: Calculus A**
**Exam. II**
**October 15, 2030**

Name: ______________________________

Class Time: __________________________

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

**Sign the pledge.** "On my honor, I have neither given nor received unauthorized aid on this Exam":

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Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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Name: ____________________________

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Multiple Choice

**1.**(5 pts.) The position of a particle moving on a straight line is given by $s(t) = \tan(t)$. What is the **acceleration** of the particle at time $t$?

(a) $4\sec(t)\tan^2(t)$

(b) $2\sec^2(t)\tan(t)$

(c) $\sec^2(t)\tan^2(t)$

(d) $2\sec(t)\tan(t)$

(e) $\sec(t)\tan^2(t) + 2\sec^3(t)$

**2.**(5 pts.) Find the linearization of $f(x) = 2xe^{x-1}$ at $x = 1$.

(a) $f(x) \approx 2(x-1) + 4$ for $x$ near 1.

(b) $f(x) \approx (2x+2)e^{x-1}(x-2) + 1$ for $x$ near 1.

(c) $f(x) \approx 4(x+2) + 1$ for $x$ near 1.

(d) $f(x) \approx (2x+2)e^{x-1}(x-1) + 2$ for $x$ near 1.

(e) $f(x) \approx 4(x-1) + 2$ for $x$ near 1.

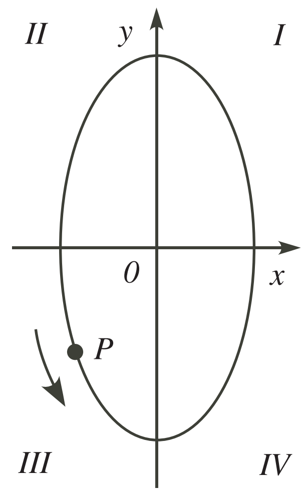
Name: ______________________________

Class Time: __________________________

**3.**(5 pts.) Consider a particle $P$ moving **counterclockwise** around the ellipse

$$4x^2 + y^2 = 5.$$

Which of the following statement is **TRUE** about $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in Quadrant II?




(a) None of the choices.

(b) $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$ when $P$ is in Quadrant II.

(c) $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$ when $P$ is in Quadrant II.

(d) $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0$ when $P$ is in Quadrant II.

(e) $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$ when $P$ is in Quadrant II.

**4.**(5 pts.) For the same Particle $P$ above, find $\frac{dx}{dt}$ at $(1, -1)$ if $\frac{dy}{dt} = 2$ units per second.

(a) $1/2$ units per second.

(b) $9/4$ units per second.

(c) $-9/4$ units per second.

(d) $-1$ unit per second.

(e) $1$ unit per second.

Name: ____________________________

Class Time: ________________________

**5.**(5 pts.) Find the third derivative $\dfrac{d^3y}{dx^3}$ if $y = e^{3x} + x^3$.

(a) $\dfrac{d^3y}{dx^3} = 3x(3x-1)(3x-2)e^{3x-3} + 6$

(b) $\dfrac{d^3y}{dx^3} = 27e^{3x}\ln 27 + 6$

(c) $\dfrac{d^3y}{dx^3} = \dfrac{e^{3x}}{27} + 6$

(d) $\dfrac{d^3y}{dx^3} = 27e^{3x} + 6$

(e) $\dfrac{d^3y}{dx^3} = e^{3x} + 6$

**6.**(5 pts.) A rectangular shape is changing its dimensions such that its length $L$ is **increasing** at 2 cm/sec and width $W$ is **decreasing** at 1 cm/sec. At what rate is the area of the rectangle changing when $L = 5$ cm and $W = 4$ cm?

(a) $-2$ cm$^2$/sec

(b) 2 cm$^2$/sec

(c) 3 cm$^2$/sec

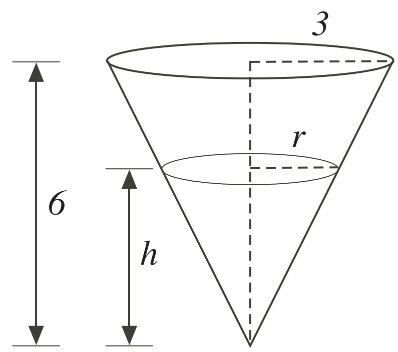
(d) $-3$ cm$^2$/sec

(e) 13 cm$^2$/sec

Name: ______________________________

Class Time: __________________________

**7.**(5 pts.) Let $V$ be the volume of water in an inverted cone of radius 3 meters and the height 6 meters. Find the volume $V$ of water in terms of the height $h$ of the water level. You may use the formula $V = \dfrac{1}{3}\pi r^2 h$.




(a) $V = \dfrac{8\pi h^3}{3}$

(b) $V = \dfrac{2\pi h^3}{3}$

(c) $V = \dfrac{\pi h^3}{6}$

(d) $V = \dfrac{\pi h^3}{12}$

(e) $V = \dfrac{\pi h^3}{24}$

**8.**(5 pts.) The intensity $I$ (in lumens) of light penetrating the water of a lake is related to the amount of sediments $s$ in the water by $I(s) = s^{-2}$. If the amount of sediments $s = e^h$ where $h$ (in meters) is the depth of the lake, what is the value of $\dfrac{dI}{dh}$ when $h = 2$ meters?

(a) $-2e^{-4}$ lumens/meter.

(b) $-e^{-4}$ lumens/meter.

(c) $1/4$ lumens/meter.

(d) $-1/4$ lumens/meter.

(e) $e^{-4}$ lumens/meter.

Name: ____________________________

Class Time: ________________________

**9.**(5 pts.) Use linear approximation to estimate the value of $f(2.1) - f(2)$ if $f(2) = 3$ and $f'(2) = -1$

(a) $f(2.1) - f(2) \approx 3.1$

(b) $f(2.1) - f(2) \approx 0.1$

(c) $f(2.1) - f(2) \approx 1.9$

(d) $f(2.1) - f(2) \approx -0.1$

(e) $f(2.1) - f(2) \approx 2.9$

**10.**(5 pts.) Find the instantaneous rate of change of $f(x) = \dfrac{\sin x}{1 + \cos x}$.

(a) $\dfrac{-\cos x - \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$

(b) $\dfrac{-1}{(1 + \cos x)}$

(c) $\dfrac{1}{(1 + \cos x)}$

(d) $\dfrac{-2}{(1 + \cos x)^2}$

(e) $\dfrac{\cos x + \cos^2 x - \sin^2 x}{(1 + \cos x)^2}$

Name: ____________________________

Class Time: ________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

**11.**(12 pts.) **(a)** Find the equation of the tangent line to the curve given by the parametric equations

$$x = \sqrt{2t+5}; \qquad y = te^{t^2-4}$$

at $t = 2$

**(b) (Not related to the above)** Find the value of $k$ such that the following function $f(x)$ is continuous at $x = 0$:

$$f(x) = \begin{cases} 3k + \dfrac{(\sin 3x)^2}{4x} & x \neq 0 \\ k + 4 & x = 0 \end{cases}$$

Carefully show all your work with limits.

Name: ____________________________

Class Time: ________________________

**12.**(12 pts.) Using logarithmic differentiation, find the derivative of the following function in terms of $x$ only.

$y = (1+x)^{x^2}$

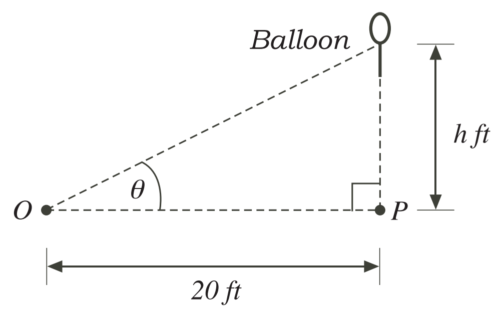
**(b) (Not related to the above)** Two cars left an intersection at the same time with one heading north at 40mph and the other heading east at 50mph. Assuming that the roads are straight and the cars keep their directions of travel, find how fast is the distance between the cars changing two hours after they have left the intersection.

Name: ____________________________

Class Time: ________________________

**13.**(12 pts.) **(a)** Find the instantaneous rate of change of $f(x) = \ln\left(\dfrac{e^x + 2}{e^x + 4}\right)$.

**(b) (Not related to the above)** A balloon is released at a point $P$, 20 feet from an observer $O$, on a hot day with still air (See figure below). If the balloon is rising **vertically** at 3 ft/sec, how fast is the angle $\theta$ of elevation at $O$ changing when $\theta = \dfrac{\pi}{4}$ radians. Your answer should contain no trigonometric functions.



Name: ______________________________

Class Time: __________________________

**14.**(12 pts.) Use implicit differentiation to find $\dfrac{dy}{dx}$ if $e^{x+y} + xy = y^3$.

**Math 10350: Calculus A**
**Exam. II**
**October 15, 2030**

Name: ______________________________

Class Time: ANSWERS

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**11.**(12 pts.) **(a)** Find the equation of the tangent line to the curve given by the parametric equations

$$x = \sqrt{2t+5}; \qquad y = te^{t^2-4}$$

at $t = 2$

One point : $t = 2$, $x = \sqrt{4+5} = 3$

$y = 2e^{4-4} = 2$

Slope at $t = 2$ :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{te^{t^2-4}(2t) + e^{t^2-4}}{\cancel{\frac{1}{2}}(2t+5)^{-1/2}\cdot \cancel{2}}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{2e^0(4) + e^0}{(4+5)^{-1/2}} = \frac{9}{1/\sqrt{9}} = 9 \times 3 = 27$$

Tangent at $t = 2$ is $y - 2 = 27(x-3)$

i.e. $y = 27x - 79$.

**(b)** Find the value of $k$ such that the following function $f(x)$ is continuous at $x = 0$:

$$f(x) = \begin{cases} 3k + \dfrac{(\sin 3x)^2}{4x} & x \neq 0 \\ k + 4 & x = 0 \end{cases}$$

Carefully show all your work with limits.

**Solution.** For $f$ to be continuous at $x = 0$, we must have that

$$\lim_{x \to 0} f(x) = f(0) = k + 4.$$

We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left( 3k + \dfrac{(\sin 3x)^2}{4x} \right) = 3k + \lim_{x \to 0} \dfrac{(\sin 3x)^2}{4x}$.

We compute $\lim_{x \to 0} \dfrac{(\sin 3x)^2}{4x} = \lim_{x \to 0} \sin 3x \cdot \dfrac{\sin 3x}{4x}$.

We know that $\sin 3x \to 0$ as $x \to 0$, and that

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{3}{4} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4}(1) = \frac{3}{4}.$$

Thus, $\lim_{x \to 0} \dfrac{(\sin 3x)^2}{4x} = (0)(\dfrac{3}{4}) = 0$.

We conclude that for $f$ to be continuous at $x = 0$, we must have $k + 4 = 3k$, or $k = 2$.

**12.** Using logarithmic differentiation, find the derivative of the following function in terms of $x$ only.

$$y = (1+x)^{x^2}$$

**Solution.** Recalling that $e^x$ and $\ln x$ are inverses of one another, we have

$$y = (1+x)^{x^2} = e^{\ln(1+x)^{x^2}} = e^{x^2\ln(1+x)}.$$

Letting $u = x^2\ln(1+x)$, we thus have that $y = e^u$. By the chain rule,

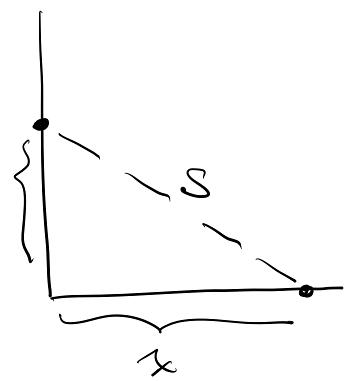
$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= \frac{d}{du}(e^u)\ \frac{d}{dx}(x^2\ln(1+x)) = e^u\ (2x\ln(1+x) + x^2(\frac{1}{1+x})) \\
&= e^{x^2\ln(1+x)}\ (2x\ln(1+x) + \frac{x^2}{1+x}) = e^{\ln(1+x)^{x^2}}\ (2x\ln(1+x) + \frac{x^2}{1+x}) \\
&= (1+x)^{x^2}\ (2x\ln(1+x) + \frac{x^2}{1+x}) = x\ (1+x)^{x^2}\ (2\ln(1+x) + \frac{x}{1+x}).
\end{aligned}$$

**Alternative:** Consider $\ln(y) = \ln(1+x)^{x^2} = x^2\ln(1+x)$.

$$\begin{aligned}
\frac{d}{dx}(\ln(y)) &= \frac{d}{dx}(x^2\ln(1+x)) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= x^2\cdot\frac{1}{1+x} + 2x\ln(1+x) \\
\frac{dy}{dx} &= y\left(\frac{x^2}{1+x} + 2x\ln(1+x)\right) \\
&= (1+x)^{x^2}\ \left(2x\ln(1+x) + \frac{x^2}{1+x}\right)
\end{aligned}$$

**(b) (Not related to the above)** Two cars left an intersection at the same time with one heading north at 40mph and the other heading east at 50mph. Assuming that the roads are straight and the cars keep their directions of travel, find how fast is the distance between the cars changing two hours after they have left the intersection.

$\frac{dx}{dt} = 50$ ; $\frac{dy}{dt} = 40$




Find $\frac{ds}{dt}$ when $t = 2$.

$s^2 = x^2 + y^2$ by Pythagorean Thm

$(s(t))^2 = (x(t))^2 + (y(t))^2$

$\frac{d}{dt}\left[(s(t))^2\right] = \frac{d}{dt}\left[(x(t))^2 + (y(t))^2\right]$

$2s(t)\frac{ds}{dt} = 2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

At $t=0$, $x=0=y$ $\Rightarrow$ ~~At~~ $t = 2$, $x = 50 \times 2 = 100$

$y = 40 \times 2 = 80$

$s = \sqrt{x^2+y^2} = \sqrt{10000 + 6400}$

$= \sqrt{16400} = 10\sqrt{\underbrace{164}_{41\times 4}} = 20\sqrt{41}$

$t=2$: $20\sqrt{41}\frac{ds}{dt} = 100(50) + 80(40)$

$\frac{ds}{dt} = \frac{5000 + 3200}{20\sqrt{41}} = \frac{20(250+160)}{20\sqrt{41}} = \frac{410}{\sqrt{41}}$

$= 10\sqrt{41}$ mph.

**13. (a)** Find the instantaneous rate of change of $f(x) = \ln\left(\frac{e^x+2}{e^x+4}\right)$.

**Solution.** We seek to find $f'(x)$. We could use the chain rule, followed by the quotient rule on $\frac{e^x+2}{e^x+4}$, but that would be a lot of work. So instead, let's remember properties of logarithms, and write

$$f(x) = \ln\left(\frac{e^x+2}{e^x+4}\right) = \ln(e^x+2) - \ln(e^x+4).$$

Now we use the chain rule to obtain

$$\begin{aligned}
f'(x) &= \frac{1}{e^x+2}\frac{d}{dx}(e^x+2) - \frac{1}{e^x+4}\frac{d}{dx}(e^x+4) \\
&= \frac{1}{e^x+2}(e^x) - \frac{1}{e^x+4}(e^x) \\
&= e^x\left[\frac{1}{e^x+2} - \frac{1}{e^x+4}\right] \\
&= \frac{2e^x}{(e^x+2)(e^x+4)}.
\end{aligned}$$

**(b)** A balloon is released at a point $P$, 20 feet from an observer $O$, on a hot day with still air (See figure below). If the balloon is rising **vertically** at 3 ft/sec, how fast is the angle $\theta$ of elevation of $O$ changing when $\theta = \pi/4$ radians? Your answer should contain no trigonometric functions.

**Solution.** Letting $h$ denote the height of the balloon, as in the figure, and letting $t$ denote time, we are given $\left.\frac{dh}{dt}\right|_{\theta=\pi/4} = 3$. We wish to find $\left.\frac{d\theta}{dt}\right|_{\theta=\pi/4}$. By basic trigonometry, we have

$$\tan\theta = \frac{h}{20},$$

and so by implicit differentiation and the chain rule,

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{20}\,\frac{dh}{dt}.$$

Thus,

$$\begin{aligned} \left.\frac{d\theta}{dt}\right|_{\theta=\pi/4} &= \frac{1}{20\sec^2\theta} \left.\frac{dh}{dt}\right|_{\theta=\pi/4} \\ &= \frac{1}{20\sec^2(\pi/4)}\ (3) \\ &= \frac{3}{(20)(\frac{2}{\sqrt{2}})^2} \\ &= \frac{3}{40}. \end{aligned}$$

**14.** Use implicit differentiation to find $\dfrac{dy}{dx}$ if $e^{x+y} + xy = y^3$.

**Solution.** Using the chain rule, we differentiate the left-hand side with respect to $x$ to get:

$$\begin{aligned}
\frac{d}{dx}(e^{x+y} + xy) &= (e^{x+y})\ \frac{d}{dx}(x+y) + \frac{d}{dx}(xy) \\
&= (e^{x+y})(1 + \frac{dy}{dx}) + (1 \cdot y + x\ \frac{dy}{dx}) \\
&= (e^{x+y})(1 + \frac{dy}{dx}) + (y + x\ \frac{dy}{dx}) \\
&= e^{x+y} + e^{x+y}\ \frac{dy}{dx} + y + x\ \frac{dy}{dx} \\
&= e^{x+y} + y + \frac{dy}{dx}(e^{x+y} + x).
\end{aligned}$$

On the right-hand side, we obtain

$$\frac{dy}{dx}(y^3) = 3y^2\ \frac{dy}{dx}.$$

Setting the two sides equal gives us

$$e^{x+y} + y + \frac{dy}{dx}(e^{x+y} + x) = 3y^2\ \frac{dy}{dx},$$

and so

$$\begin{aligned}
e^{x+y} + y &= \frac{dy}{dx}\ (3y^2) - \frac{dy}{dx}(e^{x+y} + x) \\
&= \frac{dy}{dx}\ (3y^2 - e^{x+y} - x).
\end{aligned}$$

We conclude that

$$\frac{dy}{dx} = \frac{e^{x+y} + y}{3y^2 - e^{x+y} - x}.$$

**Math 10350: Calculus A**
**Sample Exam II**
**October 18, 2009**

Name: ______________________________

Instructor: __________________________

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 11 pages of the test.

**Sign the pledge.** "On my honor, I have neither given nor received unauthorized aid on this Exam":

________________________________________________________________

Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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Name: ______________________________

Instructor: __________________________

Multiple Choice

**1.**(5 pts.) What is the derivative of $\sec^2(5x)$?

(a) $2\sec(5x)\tan(5x)$

(b) $10\sec^2(5x)\tan(5x)$

(c) $10\sec(5x)$

(d) $5\tan(5x)$

(e) $10\tan(5x)$

**2.**(5 pts.) Find the linear approximation of the function $f(x) = \sqrt[5]{x} + 3$ at $x = -1$.

(a) $f(x) \approx \frac{1}{5}(x-1) - 2$ for $x$ near $-1$.

(b) $f(x) \approx \frac{x^{-4/5}}{5}(x+1) + 2$ for $x$ near $-1$.

(c) $f(x) \approx \frac{1}{5}(x-2) - 1$ for $x$ near $-1$.

(d) $f(x) \approx \frac{x^{-4/5}}{5}(x-1) - 2$ for $x$ near $-1$.

(e) $f(x) \approx \frac{1}{5}(x+1) + 2$ for $x$ near $-1$.

Name: ______________________________

Instructor: __________________________

**3.**(5 pts.) The volume of a spherical snow ball is **decreasing** at a rate of $\pi$ cm$^3$ per second when its radius is 5 cm. What is the rate of change of the radius at this moment? You may use the formula $v = \frac{4}{3}\pi r^3$.

(a) $-1/100$ cm/sec

(b) $1/20$ cm/sec

(c) $1/200$ cm/sec

(d) $-1/500$ cm/sec

(e) $-1/20$ cm/sec

**4.**(5 pts.) Consider the motion of a particle moving on the circle $x^2 + y^2 = 10$. Find the value of $\frac{dy}{dt}$ when the particle is located at $(1, 3)$ and $\frac{dx}{dt} = -2$.

(a) $1/3$

(b) $-2/3$

(c) $-1/3$

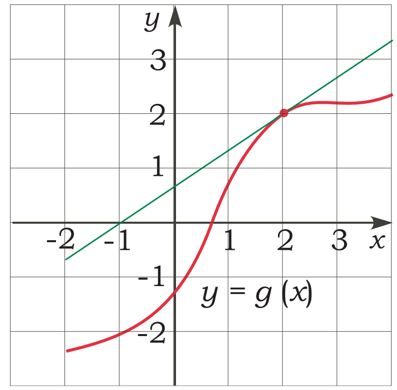
(d) $2/3$

(e) $2$

Name: ______________________________

Instructor: __________________________

**5.**(5 pts.) The graph of the function $g(x)$ is given below. Find the derivative of $f(x) = g(x^2 + 1)$ at $x = 1$.




(a) $f'(1) = 4$

(b) $f'(1) = 4/3$

(c) $f'(1) = 2$

(d) $f'(1) = 2/3$

(e) $f'(1) = 3/2$

**6.**(5 pts.) The position function $s(t)$ at time $t$ of a device floating in a pool measured from the bottom of the pool is given by:

$$s(t) = 5 + \sin(t) - \cos(t).$$

Find the instantaneous velocity of the device at time $t = \dfrac{\pi}{3}$.

(a) $5+\dfrac{1-\sqrt{3}}{2}$

(b) $\dfrac{-1-\sqrt{3}}{2}$

(c) $\dfrac{-1+\sqrt{3}}{2}$

(d) $\dfrac{1-\sqrt{3}}{2}$

(e) $\dfrac{1+\sqrt{3}}{2}$

Name: ______________________________

Instructor: __________________________

**7.**(5 pts.) Find the value of $k$ such that the following function is continuous at $x = 0$:

$$f(x) = \begin{cases} 1 + \dfrac{\sin 5x}{\sin 7x} & x \neq 0 \\ k & x = 0 \end{cases}$$

(a) 2

(b) 12/5

(c) 5/7

(d) 12/7

(e) 7/5

**8.**(5 pts.) Suppose that

$$f(1) = 2, \qquad f(2) = 1, \qquad g(1) = 4, \qquad g(2) = 3,$$

$$f'(1) = 6, \qquad f'(2) = 5, \qquad g'(1) = 8, \qquad g'(2) = 7.$$

Let $h(x) = g(f(x))$. What is $h'(1)$?

(a) 42

(b) 12

(c) 40

(d) 14

(e) 48

Name: ______________________________

Instructor: __________________________

**9.**(5 pts.) Consider the function

$$f(x) = \begin{cases} \dfrac{x}{\sin(kx)} & \text{if } x < 0 \\ 2x+5 & \text{if } x \geq 0 \end{cases}$$

Find the value of $k$, if it exists, so that $f(x)$ is continuous at $x = 0$.

(a) $5$ (b) $\frac{1}{2}$ (c) $2$

(d) $\frac{1}{5}$ (e) Does not exist.

**10.**(5 pts.) If $f(x) = \sqrt{2x+7}$, then $f'''(x) = ?$

(a) $-\dfrac{3}{8\sqrt{(2x+7)^5}}$

(b) $\dfrac{1}{\sqrt{2x+7}}$

(c) $\dfrac{3}{\sqrt{(2x+7)^5}}$

(d) $\dfrac{3}{8\sqrt{(2x+7)^5}}$

(e) $-\dfrac{3}{\sqrt{(2x+7)^5}}$

Name: ____________________________

Instructor: __________________________

**11.**(5 pts.) What is the derivative of $\tan(x^3+1)$?

(a) $\sec^2(x^3+1)+\tan(3x^2)$

(b) $-\cot(x^3+1)$

(c) $3x^2\sec^2(x^3+1)$

(d) $\sec^2(x^3+1)$

(e) $3x^2\sec(x^3+1)\tan(x^3+1)$

**12.**(5 pts.) The pilot of the helicopter flying over the sea observed that **the area** of an oil slick caused by a sunken ship is growing a rate of $4\pi$ square kilometers per hour. Assuming that the oil slick is **circular**, at what rate is **the radius** of the oil slick growing when its radius is 1/2 kilometers?

(a) 2 km/hr

(b) $\frac{\pi}{4}$ km/hr

(c) $2\pi$ km/hr

(d) $4\pi^2$ km/hr

(e) 4 km/hr

Name: ______________________________

Instructor: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

**13.**(10 pts.) (a) Consider the curve given by

$$y^3 - 2xy = 5 - x^3.$$

Use **implicit differentiation** to find $\dfrac{dy}{dx}$.

(b) For a certain curve $C$: $\dfrac{dy}{dx} = \dfrac{x - y + 2}{x + y + 1}$.

If the point $(1, -1)$ is on the curve, find the equation of the tangent line to $C$ at $(1, -1)$.

Name: ____________________________

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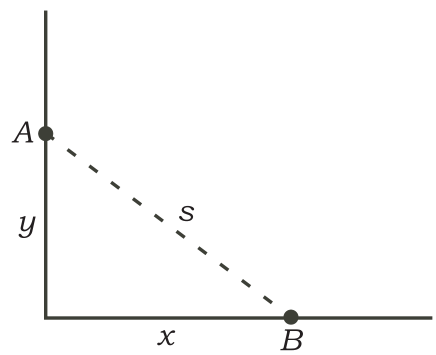
**14.**(10 pts.)
**14A.** Find the derivative of $y = (1+x)^{e^{2x}}$ using logarithmic differentiation.

**14B (Not related to above).** Water is following in an inverted cone of radius 1 m and height 2 m at a rate of 0.5 m$^3$/min. How fast is the **radius** of the water surface growing when its diameter is 1m?

Name: ______________________________

Instructor: __________________________

**15.**(10 pts.) Cyclist $A$, approaching a right angled intersection from the north, is chasing cyclist $B$ who has turned the corner and is now moving straight east. When cyclist $A$ is 3 meters north of the corner, cyclist $B$ is 4 meters east of the corner traveling at a speed of 10 meters per second. Given that the distance between $A$ and $B$ is **increasing** at a rate of 2 meters per second, find the speed of cyclist $A$.



Name: ______________________________

Instructor: __________________________

**16.**(10 pts.) **16A.** Consider the function $f(x) = \sqrt[3]{x}$.

**i.** Find the linear approximation of $f(x)$ at $x = -8$.

**ii.** Using (a), estimate the value of $\sqrt[3]{-7.8}$

**16B. (Not related to above)** Air is pumped into a perfectly spherical balloon of radius $r$. Using linear approximation, estimate the **change** in the volume of the balloon when the radius changes from 3 cm to 3.05 cm.

You may use the formula $V = \dfrac{4\pi r^3}{3}$.

**Math 10350: Calculus A**
**Sample Exam II**
**October 18, 2009**

Name: ____________________

Instructor: ANSWERS

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- No calculators.
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Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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**13.** **(a)** Consider the curve given by

$$y^3 - 2xy = 5 - x^3.$$

Use **implicit differentiation** to find $\dfrac{dy}{dx}$.

**Solution.** Differentiating the left-hand side with respect to $x$ gives us

$$\begin{aligned}\frac{d}{dx}(y^3 - 2xy) &= \frac{d}{d(y)}(y^3)\ \frac{dy}{dx} - (2(1)y + 2x\frac{dy}{dx}) \\ &= 3y^2\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} \\ &= (3y^2 - 2x)\frac{dy}{dx} - 2y,\end{aligned}$$

whereas on the right-hand side we get

$$\frac{d}{dx}(5 - x^3) = -3x^2.$$

Setting the two sides and equal and solving for $\dfrac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2x}.$$

A cool derivative!

**(b)** For a certain curve $C$: $\dfrac{dy}{dx} = \dfrac{x - y + 2}{x + y + 1}$. If the point $(1, -1)$ is on the curve, find the equation of the tangent line to $C$ at $(1, -1)$.

**Solution.** We have

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{1 - (-1) + 2}{1 + (-1) + 1} = \frac{4}{1} = 4.$$

The equation of the tangent line to $C$ at the point $(1, -1)$ is thus given by

$$y = 4(x - 1) + (-1) = 4x - 5.$$

**14. (a)** Find the derivative of $y = (1+x)^{e^{2x}}$ using logarithmic differentiation.

**Solution.**

$$\ln y = \ln(1+x)^{e^{2x}}$$

$$\ln y = e^{2x} \ln(1+x)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[e^{2x}\ln(1+x)\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = e^{2x}\frac{1}{(1+x)} + 2e^{2x}\ln(1+x)$$

$$\frac{dy}{dx} = y\left[\frac{e^{2x}}{(1+x)} + 2e^{2x}\ln(1+x)\right]$$

$$\frac{dy}{dx} = (1+x)^{e^{2x}}\left[\frac{e^{2x}}{(1+x)} + 2e^{2x}\ln(1+x)\right]$$

**(b)** Water is flowing into an inverted cone of radius 1 m and height 2 m at a rate of 0.5 m$^3$/min. How fast is the **radius** of the water surface growing when its diameter is 1 m?

**Solution.** Let $r$ be the radius of the water surface, $h$ the water level, and $V$ the volume. Then

$$V = \frac{1}{3}\pi r^2 h.$$

By similarity of triangles, we thus have the relationship

$$\frac{h}{r} = \frac{\text{height of cone}}{\text{radius of cone}} = \frac{2 \text{ m}}{1 \text{ m}} = 2,$$

and so $h = 2r$. Therefore,

$$V = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3.$$

Letting $t$ denote time in minutes, we are given that $\dfrac{dV}{dt} = 0.5$, or

$$0.5 = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{2}{3}\pi r^3\right) = 2\pi r^2 \frac{dr}{dt}.$$

Thus,

$$\frac{dr}{dt} = \frac{0.5}{2\pi r^2} = \frac{1}{4\pi r^2}.$$

Now we wish to find $dr/dt$ when the diameter of the water surface is 1 m, or in other words, when the radius if 0.5 m. This is therefore

$$\left.\frac{dr}{dt}\right|_{r=0.5} = \frac{1}{4\pi(0.5)^2} = \frac{1}{4\pi(\frac{1}{2})^2} = \frac{1}{\pi}.$$

**15.** Cyclist $A$, approaching a right angled intersection from the north, is chasing cyclist $B$ who has turned the corner and is now moving straight east. When cyclist $A$ is 3 meters north of the corner, cyclist $B$ is 4 meters east of the corner traveling at a speed of 10 meters per second. Given that the distance between $A$ and $B$ is **increasing** at a rate of 2 meters per second, find the speed of cyclist $A$.

**Solution.** Let $D(t)$ denote the distance between $A$ and $B$, where $t$ is the time in seconds. Let $A(t)$ denote the position of $A$ at time $t$ to the north of the origin, and $B(t)$ the position of $B$ to the east of the origin. Notice that $A(t)$ and $B(t)$ are also the distances of $A$ and $B$, respectively, from the origin. So, by the Pythagorean theorem, we have

$$D(t)^2 = A(t)^2 + B(t)^2.$$

Since distance is never negative, we thus have

$$D(t) = \sqrt{A(t)^2 + B(t)^2}.$$

We want to find the speed of $A$, or in other words, $\frac{dA}{dt}$, at the time $t$ when $A(t) = 3$ m and $B(t) = 4$ m. Rewriting the first equation above gives

$$A(t)^2 = D(t)^2 - B(t)^2,$$

so by implicit differentiation,

$$2A(t)\frac{dA}{dt} = 2D(t)\frac{dD}{dt} - 2B(t)\frac{dB}{dt},$$

so

$$\frac{dA}{dt} = \frac{2D(t)\frac{dD}{dt} - 2B(t)\frac{dD}{dt}}{2A(t)} = \frac{D(t)\frac{dD}{dt} - B(t)\frac{dD}{dt}}{A(t)}.$$

Now at the time in question, we are given that

$$\frac{dD}{dt} = 2 \text{ m/sec.}$$

and that

$$\frac{dB}{dt} = 10 \text{ m/sec,}$$

and we can figure out that

$$D(t) = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5.$$

Thus, at this time, we have

$$\frac{dA}{dt} = \frac{(5)(2) - (4)(10)}{3} = \frac{10 - 40}{3} = -10 \text{ m/s.}$$

So $A$ is going at a speed of 10 m/s in the direction of the origin, as indicated by the minus sign.

**16. (a)** Consider the function $f(x) = \sqrt[3]{x}$.

**(i)** Find the linear approximation of $f(x)$ at $x = -8$.

**Solution.** We have $f(x) = x^{1/3}$ so $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(-8) = \frac{1}{3}(-8)^{-2/3} = \frac{1}{3}(-2)^{-2} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}$. The equation for the linear approximation to $f$ at $x = -8$ is thus

$$L(x) = \frac{1}{12}(x+8) - 2.$$

**(ii)** Using (i), estimate the value of $\sqrt[3]{-7.8}$.

**Solution.** We have that

$$f(-7.8) \approx L(-7.8) = \frac{0.2}{12} - 2 = \frac{1}{60} - 2 = -\frac{119}{60}.$$

**(b)** Air is pumped into a perfectly spherical balloon of radius $r$. Using calculus, estimate the change in the volume of the balloon when the radius changes from 3 cm to 3.05 cm. You may use the formula $V = \dfrac{4\pi r^3}{3}$.

**Solution.** The change in V as $r$ increases from 3 to 3.05 is given by

$$\Delta V \approx V'(3)\Delta r = V'(3)(3.05 - 3) = V'(3)(0.05).$$

From the formula for $V$, we have

$$V'(r) = 4\pi r^2,$$

and so $V'(3) = 4\pi(3)^2 = 36\pi$. Thus

$$\Delta V = (36\pi)(0.05) = 1.8\pi.$$