Math 10260 Exam 3 Review Activity

1. Consider the differential equation $\frac{dy}{dt} = g(y)$ where the graph of $g(y)$ is given in Figure 1. (a) Find the equilibrium solutions and determine their stability. (b) Use graphical method to sketch the graph of the solution for the given initial value; you should clearly show all important features like concavity and asymptotic behavior:
$y(0) = -3, \quad y(0) = -1, \quad y(0) = 0, \quad y(0) = 1, \quad y(0) = 4$

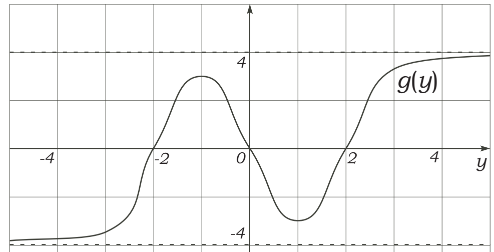

Figure 1

2. The size $p(t)$ of a certain population (in hundreds) of hedgehog is modeled by
$$\frac{dp}{dt} = \frac{1}{50}(-p^3 + 19p^2 - 60p).$$
(a) Find the threshold level of this population, (b) Find the carrying capacity, (c) For what values of $p(0)$ (initial population) will the population be headed for extinction? (Ans: (a) 4, (b) 15, (c) $0 < p(0) < 4$)

3. Find the 4th-degree Taylor polynomial of $f(x) = \ln(x-2)$ about $a = 3$. Use it to estimate $\ln(1.2)$.
(Ans:$(x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} - \frac{(x-3)^4}{4} = \sum_{k=1}^{4}(-1)^{k+1}\frac{(x-3)^k}{k}$; $\ln(1.2) \approx 0.1822\bar{6}$)

4. The demand function for a certain type of cake at a bakery is given by $p = f(x)$ where p is the price in units of ten dollars, and x is the monthly demand of the cake in units of tens. The cost of producing x cakes in a month is given by the function $C(x) = x^2/4$ in hundreds of dollars. (a) Write down the **profit function** $P(x)$ (in hundreds of dollars); (b) Find the 2nd-degree Taylor polynomial of $P(x)$ about 2 if $f(2) = 4$, $f'(2) = -1$, and $f''(2) = 1/2$. (Ans: $7 + (x-2) - \frac{3}{4}(x-2)^2$)

5. Let $y(t)$ be the solution to the initial value problem $y' = y^3 - 2y, \quad y(-1) = 1$.
(a) Find $P_3(t)$, the 3rd-degree Taylor polynomial of $y(t)$ about -1. (Ans: $1 - (t+1) - \frac{1}{2}(t+1)^2 + \frac{5}{6}(t+1)^3$)
(b) Use (a) to approximate $y(-1.1)$ (Ans: $P_3(-1.1) = 1.09$)

6. Write down the summation formula for a geometric series, stating clearly when the series converges. Use a geometric series to write $0.245454545\cdots$ as a fraction. State clearly the first term and the common ratio of the series you use. (Ans: 27/110)

7. Consider newly incorporated business in a certain area. Data indicates that there is a 59% chance that a business fails within the first year, a 31% chance that a business is its owners first venture, and a 71% chance that a business fails during the first year **or** that it is its owners first venture. Let F be the event that a business fails within the first year and V be the event that a business is its owners first venture.
(a) Find the chance that a business fails during the first year **and** that it is its owners first venture.
(b) Compute the probability for a business that failed during the first year is its owner's first venture.
(c) Are F and V independent events? Give reason. (Ans: (a) 0.19, (b) $P(V|F)$)

8. A bowl contains ten balls, of which three are white and seven are red. Three balls are drawn without replacement from the bowl. (a) What is the probability that the first two are white and the third is red? What about first red and last two white? (b) What is the probability that at least two balls are white? (Ans: (a) 7/120, (b) 11/60)

9. A box contains three fair coins and one coin with two heads. Suppose that one of the coins is selected at random and then tossed. Let A be the event that the coin is fair, and H be the event that the toss results in a head. Compute $P(A|H)$. Write in complete sentence what probability have you computed. (Ans: 3/5)

10. The 5th-degree Taylor polynomial of $f(x)$ about 1 is $3 + (x-1) - \frac{1}{3}(x-1)^2 + \frac{1}{10}(x-1)^5$. Write down the values of $f(1)$, $f''(1)$, $f'''(1)$, and $f^{(5)}(1)$. (Ans: $f(1) = 3$, $f''(1) = -2/3$, $f'''(1) = 0$, $f^{(5)}(1) = 12$)

11. Imagine yourself in the following situation. You have already gotten that great job and you are thinking of establishing a perpetual fund that will provide an annual scholarship in your name to a worthy Notre Dame undergraduate student. As a hedge against inflation, you would like the size of the scholarship to increase by 4% every year, starting with \$8,000 at the end of the first year. You endow the fund by investing a fixed amount that will earn an annual interest rate of 7% compounded continuously. (a) Write down the **present value** of the first scholarship, second scholarship, third scholarship, and forth scholarship. (b) How large must your endowment be? (Ans: (b) \$246,091.90)

12. Customers at Pizza Palace order pizzas with different number of toppings. The table below gives the probability of the number of toppings on the pizza ordered for a randomly selected customer. Complete the table. What is the probability that a customer orders at least two topping? You may compute this in two ways. Could you see both? (Ans: 0.05, 0.55)

Number of Toppings	0	1	2	3	4 or more
Probability	0.15	0.30	0.35	0.15	

13. A patient is injected once a day with 20 units of a certain drug. Suppose this drug is eliminated by the body exponentially with any single injection leaving an amount of $20e^{-0.7t}$ remaining after t days. Use an infinite geometric series to approximate the number of units of the drug remaining in the patient's system after a very long time. Assuming that the measurement is done **before** each injection. (Ans: 19.7287)

14. The Taylor series of e^x about 0 is given as follows: $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ for all real number x.

    (a) Find the 3rd-degree Taylor polynomial of e^x about 0.

    (b) Use the result in Part (a) to estimate $e^{0.2}$. (Ans: 1.22133)

    (c) Write down the error of your estimate in Part (b) as an infinite series.

    (d) Use the 4th degree Taylor polynomial for $e^{-x^2/2}$ about 0 to estimate $\int_0^1 e^{-x^2/2} dx$. (Ans: 0.858333)

15. The number $y(t)$, in thousands, of pine trees t years after year 2000 in a 100 sq. miles region is approximately modeled by the differential equation

$$\frac{dy}{dt} = 1 + t - 0.1y^2$$

If the number of trees is 2 thousand in the year of 2000, use Euler's Method with $\Delta t = 2$ to estimate the number of trees in the region in the year of 2006. (Ans: 6.921 thousand)

16. Figure 1 shows the directional field for $\frac{dy}{dt} = g(y)$. Which of the following statements is true? (Ans: a)

    (a) A stable equilibrium solution at $y = 3$ and an unstable solution at $y = 0$.

    (b) An unstable equilibrium solution at $y = 3$ and a stable solution at $y = 0$.

    (c) Two stable equilibrium solutions at $y = 3$ and $y = 0$.

    (d) Two unstable equilibrium solutions at $y = 3$ and $y = 0$.

    (e) No equilibrium solutions.

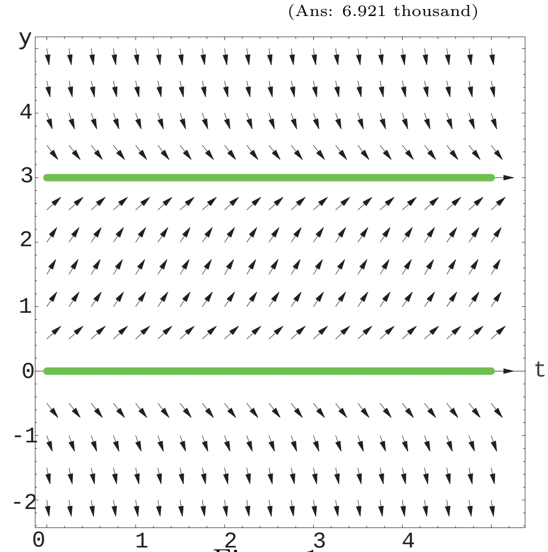

Figure 1

17. Which of the follow differential equations has the direction field shown in Figure 1

    (a) $\frac{dy}{dt} = y(y-4)$ (b) $\frac{dy}{dt} = (y-1)(y+3)$ (c) $\frac{dy}{dt} = y(y-3)$
    (d) $\frac{dy}{dt} = y^2(3-y)$ (e) $\frac{dy}{dt} = y(3-y)$ (Ans: e)

18. Squeezed between a global economy that demands increased productivity and a technology fueled entertainment machine that provides non-stop diversions Americans keep getting less and less sleep. Assume that available data indicate that on average Americans get about 6.5 hours of sleep per day, now. Also, assume that from now on the number of hours of sleep N per day is modeled by the differential equation $\frac{dN}{dt} = -0.4 \ln \frac{N}{6}$, where t is measured in years. Sketch the graph of $N(t)$, and compute its limiting value. (Ans: $\lim_{t\to\infty} N(t) = 6$)