

Math 108 Exam 3 Solutions - Spring 2005

1. $p(5) \approx p(0) + p'(0)\Delta t = 0.4 + [(0.1) \cdot (0.4) \cdot (1 - 0.4)] \cdot 5 = 0.52$

$$p(10) \approx p(5) + p'(5)\Delta t = 0.52 + [(0.1) \cdot (0.52) \cdot (1 - 0.52)] \cdot 5 = 0.6448$$

2. According to the plot of the direction field, $y = 2$ is a unstable solution and $y = -1$ is an stable solution.

3. Since $y = 1$ and $y = 0$ are equilibrium solutions, $g(1) = g(0) = 0$. Furthermore, since $y = 1$ is stable, $g'(1) < 0$ and since $y = 0$ is unstable, $g'(0) > 0$.

If $g(y) = y(1 - y) = y - y^2$, then $g'(y) = 1 - 2y$. $g'(1) = -1 < 0$ and $g'(0) = 1 > 0$. Therefore, the matching differential equation is $y' = y(1 - y)$.

4. $g(p) = -\ln(\frac{p}{150})$. Since $g(150) = 0$, $p = 150$ is an equilibrium solution. By using chain rule, $g'(p) = -\frac{1}{p}$. $g'(150) = -\frac{1}{150} < 0$ implies that it is stable. Therefore, it is the limiting value.

5. Let us compute derivatives of $f(x) = e^{5x}$.

$$f(x) = e^{5x} \Rightarrow f(0) = 1$$

$$f'(x) = 5e^{5x} \Rightarrow f'(0) = 5$$

$$f''(x) = 5^2 e^{5x} \Rightarrow f''(0) = 5^2$$

$$f'''(x) = 5^3 e^{5x} \Rightarrow f'''(0) = 5^3$$

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3, \\ &\Rightarrow = 1 + 5x + \frac{5^2}{2}x^2 + \frac{5^3}{3!}x^3. \end{aligned}$$

6. By factoring $\frac{3}{4}$ we obtain, $\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots = \frac{3}{4}[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots]$ Since $\frac{3}{4} < 1$, we can use the geometric formula. So we have $= \frac{3}{4}[\frac{1}{1 - \frac{3}{4}}] = \frac{3}{4} \times 4 = 3$

7. Let us compute derivatives of $f(x) = \frac{4}{1+x}$.

$$f(x) = \frac{4}{1+x} \Rightarrow f(0) = 4$$

$$f'(x) = -\frac{4}{(1+x)^2} \Rightarrow f'(0) = -4$$

$$f''(x) = \frac{4 \times 2}{(1+x)^3} \Rightarrow f''(0) = 4 \times 2$$

$$f'''(x) = -\frac{4 \times 2 \times 3}{(1+x)^4} \Rightarrow f'''(0) = -4 \times 2 \times 3$$

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \dots \\ &\Rightarrow = 4 - 4x + \frac{4 \times 2}{2}x^2 - \frac{4 \times 2 \times 3}{3!}x^3 \dots \\ &= 4 - 4x + 4x^2 - 4x^3 \dots \end{aligned}$$

8. The amount of drug remaining in the patient's system

= amount of drug injected today + amount of drug injected yesterday + amount of drug injected day before yesterday + ...

$$= 50e^{-0.2} + 50e^{-0.2 \cdot 2} + 50e^{-0.2 \cdot 3} + \dots$$

$$= 50e^{-0.2}(1 + e^{-0.2} + e^{-0.2 \cdot 2} + e^{-0.2 \cdot 3} + \dots)$$

$$= \frac{50e^{-0.2}}{1 - e^{-0.2}}$$

9. There are four possible ways to have to top face to add up to 5; (1, 4), (2, 3), (3, 2), (4, 1) out of 36 possible outcomes. Therefore the probability for rolling a pair of dice to have sum of five is $\frac{4}{36} = \frac{1}{9}$.

10. Since $P(E \cup F) = P(E) + P(F) - P(E \cap F)$,

$$P(F) = P(E \cup F) - P(E) + P(E \cap F) = 0.7 - 0.4 + 0.2 = 0.5.$$

$$11. k(4) \approx k(0) + k'(0) \cdot \Delta t = 2.25 + [(0.3)(\sqrt{2.25}) - (0.15)(2.25)] \cdot 4 = 2.7.$$

$$k(8) \approx k(4) + k'(4) \cdot \Delta t = 2.7 + [(0.3)(\sqrt{2.7}) - (0.15)(2.7)] \cdot 4 = 3.0516.$$

$$k(12) \approx k(8) + k'(8) \cdot \Delta t = 3.0516 + [(0.3)(\sqrt{3.0516}) - (0.15)(3.0516)] \cdot 4 = 3.3168.$$

12. $g(y) = y(4 - y) = 0$. So $y = 0$ and $y = 4$ are two equilibrium solutions. $g'(y) = 4 - 2y$. $g'(0) = 4 > 0$ implies that $y = 0$ is unstable. Furthermore $g'(4) = -4 < 0$ implies $y = 4$ is stable.

For concavities, we need to check the second derivative $\frac{d^2y}{dt^2} = g(y)g'(y) = y(4 - y)(4 - 2y)$. For finding the inflection line, we set $4 - 2y = 0$. Hence $y = 2$.

Therefore, by checking signs of the second derivative,

- The solution is concave up for $0 < y < 2$ and $y > 4$.
- The solution is concave down for $y < 0$ and $2 < y < 4$.

13. (a) Let us compute derivatives of $f(x) = \ln(1 + x)$.

$$f(x) = \ln(1 + x) \Rightarrow f(0) = 0 \qquad p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$f'(x) = (1 + x)^{-1} \Rightarrow f'(0) = 1 \qquad \Rightarrow \qquad = x - \frac{x^2}{2}$$

$$f''(x) = -(1 + x)^{-2} \Rightarrow f''(0) = -1$$

$$\text{Therefore } \int_0^1 \ln(1 + x) dx \approx \int_0^1 x - \frac{x^2}{2} dx = \left[\frac{x^2}{2} - \frac{x^3}{6} \right]_0^1 = \frac{1}{3}.$$

$$(b) \text{ Since } y' = 3t - y, y'(0) = 3(0) - y(0) = 0 - 5 = -5.$$

$$y'' = 3 - y'. \text{ Therefore } y''(0) = 3 - y'(0) = 3 - (-5) = 8.$$

$$\text{So we have } y(t) \approx y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 = 5 - 5t + 4t^2.$$

$$\text{Hence } y(0.2) \approx 5 - 5(0.2) + 4(0.2)^2 = 4.16.$$

14. To get \$10,000 at the end of the first year, you have to deposit $10000e^{-0.1(1)}$ now.

To get $\$10,000 + 4\% = \$10000(1.04)$ at the end of the second year, you have to deposit $10000(1.04)e^{-0.1(2)}$ now.

To get $\$10,000(1.04) + 4\% = \$10000(1.04)^2$ at the end of the third year, you have to deposit $10000(1.04)^2e^{-0.1(3)}$ now.

...

Therefore, to have this fund forever, you have to deposit

$$10000e^{-0.1(1)} + 10000(1.04)e^{-0.1(2)} + 10000(1.04)^2e^{-0.1(3)} + \dots$$

$$= 10000e^{-0.1(1)}(1 + (1.04)e^{-0.1(1)} + (1.04)^2e^{-0.1(2)} + \dots) = \frac{10000e^{-0.1(1)}}{1 - (1.04)e^{-0.1}}$$

15. $P(E) = 0.5 + 0.25 = 0.75$, $P(F) = 0.25 + 0.05 = 0.3$
 $P(E \cap F) = P(s_3) = 0.25$, $P(E \cup F) = P(s_2, s_3, s_4) = 0.5 + 0.25 + 0.05 = 0.8$.