Math 108 Exam 2 Solutions - Spring 2005

1. The distance is equal to $\sqrt{(-3+1)^2+(0+2)^2+(6-4)^2} = \sqrt{2^2+2^2+2^2} = \sqrt{12}$. Answer: (c).

2. By the definition of partial derivative, it is equal to $\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = y\left(x^2+y^2\right)^{-\frac{1}{2}}$. Answer: (b).

3. The equation of the tangent plane at $(1,0)$ is given by $z = f(1,0) + \frac{\partial f}{\partial x}(1,0)\cdot(x-1) + \frac{\partial f}{\partial y}(1,0)\cdot(y-0)$ or, $z = 2+(-1)(x-1)+3y$, or $z=-x+3y+3$. Answer: (a).

4. A level curve of $f(x,y) = y - x^2 + 2x$ of height z_0 is given by $y - x^2 + 2x = z_0$, or $y = x^2 - 2x + z_0$, so it is a parabola which opens up. Answer: (e).

5. The linear approximation of $f(x,y)$ at $(3,2)$ is given by $f(x,y) \sim f(3,2) + \frac{\partial f}{\partial x}(3,2)\cdot(x-3) + \frac{\partial f}{\partial y}(3,2)\cdot(y-2)$ or, $f(x,y) \sim 1+(x-3)+(-2)(y-2)$. In particular, $f(3.1,1.9) \sim 1+(3.1-3)+(-2)(1.9-2) = 1.3$. Answer: (d).

6. The first partial derivatives are given by $\frac{\partial f}{\partial x} = 6x-6y$ and $\frac{\partial f}{\partial y} = 3y^2-6x-9$. Solving $6x-6y=0$ and $3y^2-6x-9=0$, we get $x=y=-1$ or $x=y=3$. Answer: (e).

7. By direct computation, $\frac{\partial f}{\partial x} = e^{y^3}$ and $\frac{\partial^2 f}{\partial x \partial y} = 3y^2 e^{y^3}$. Answer: (a).

8. We have $D(1,-1) = (-3)(-6)-4^2 = 2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(1,-1) = -3 < 0$. Therefore, by the second derivative test, $f(x,y)$ has a local maximum at $(1,-1)$. Answer: (a).

9. The error function is given by $E(a,b) = (a\cdot 2+b-0)^2 + (a\cdot(-1)+b-1)^2 + (a\cdot 1+b-2)^2$, or $E(a,b) = (2a+b)^2+(-a+b-1)^2+(a+b-2)^2$. Answer: (c).

10. Answer: (b).

11. The equation of a plane passing through $(1,3,2)$ is given by $z-2 = a(x-1)+b(y-3)$. To determine a and b, we notice that $(3,-1,6)$, $(5,0,0)$ are also on the plane. Thus we have $6-2 = a(3-1)+b(-1-3)$ and $0-2 = a(5-1)+b(0-3)$, or $4 = 2a-4b$ and $-2 = 4a+-3b$. By eliminating either a or b, we get $a=b=-2$. So the equation of the plane is given by $z-2 = -2(x-1)-2(y-3)$, or $z = -2x-2y+10$.

12(a). The first partial derivatives are given by $\frac{\partial f}{\partial x} = 3x^2y+24x$ and $\frac{\partial f}{\partial y} = x^3-8$. Solving $3x^2y+24x=0$ and $x^3-8=0$, we get $x=2$ and $y=-4$.

12(b). $\frac{\partial^2 f}{\partial x^2} = 6x-6$, $\frac{\partial^2 f}{\partial y^2} = 6x-6$, and $\frac{\partial^2 f}{\partial x \partial y} = 6y$. Thus $D(x,y) = (6x-6)\cdot(6x-6)-(6y)^2 = 36(x-1)^2-36y^2$. Since $D(1,1) = -36 < 0$, we conclude that $(1,1)$ is a saddle point. Since $D(1,-1) = -36 < 0$, we conclude that $(1,-1)$ is a saddle point. Since $D(0,0) = 36 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) = -6 < 0$, we conclude that $(0,0)$ is a local maximum point. Finally, since $D(2,0) = 36 > 0$ and $\frac{\partial^2 f}{\partial x^2}(2,0) = 6 > 0$, we conclude that $(2,0)$ is a local minimum point.

13(a). The linear approximation of $f(x,y) = \ln(x-3y)$ at $(7,2)$ is given by $\ln(x-3y) \sim f(7,2) + \frac{\partial f}{\partial x}(7,2)\cdot(x-7) + \frac{\partial f}{\partial y}(7,2)\cdot(y-2)$. On the other hand, $\frac{\partial f}{\partial x} = \frac{1}{x-3y}$ and $\frac{\partial f}{\partial y} = \frac{-3}{x-3y}$. Thus $\frac{\partial f}{\partial x}(7,2) = 1$ and $\frac{\partial f}{\partial y}(7,2) = -3$. Of course, $f(7,2) = \ln 1 = 0$. It follows that $\ln(x-3y) \sim (x-7)-3(y-2)$.

13(b). Let x be the number of item X, y the number of item Y, and $P(x,y)$ the profit function.

We know that $P(150,200) = 40,000$, $\frac{\partial P}{\partial x}(150,200) = 100$ and $\frac{\partial P}{\partial y}(150,200) = 150$. Thus the linear approximation of $P(x,y)$ at $(150,200)$ is given by $P(x,y) \sim 40,000 + 100(x-150) + 150(y-200)$. In particular, $P(160,190) \sim 40,000+100(160-150)+150(190-200) = 39,500$.

14(a). The profit function $P(x,y) = R(x,y)-C(x,y) = x(25-0.2x)+y(10-0.05y)-15-0.2(x+y)$, or $P(x,y) = -0.2x^2 - 0.05y^2 + 24.8x + 9.8y - 15$.

14(b). The first partial derivatives are given by $\frac{\partial P}{\partial x} = -0.4x + 24.8$ and $\frac{\partial P}{\partial y} = -0.1y + 9.8$. Solving $-0.4x + 24.8 = 0$ and $-0.1y + 9.8 = 0$, we get $x = 62$ or $y = 98$.

14(c). $p_1 = 25 - 0.2 \times 62 = 12.6$ and $p_2 = 10 - 0.05 \times 98 = 5.1$.

15(a). The question is reduced to find the maximum of $P(K,L) = 100K^{\frac{1}{3}}L^{\frac{2}{3}}$ subject to the constraint $Q(K,L) = 20K + 10L - 120,000 = 0$.

15(b). First we have $\frac{\partial P}{\partial K} = 100 \cdot \frac{1}{3} \cdot K^{-\frac{2}{3}}L^{\frac{2}{3}}$ and $\frac{\partial P}{\partial L} = 100K^{\frac{1}{3}} \cdot \frac{2}{3} \cdot L^{-\frac{1}{3}}$. Also $\frac{\partial Q}{\partial K} = 20$, $\frac{\partial Q}{\partial L} = 10$. By Lagrange multiplier method, we get three equations:
$100 \cdot \frac{1}{3} \cdot K^{-\frac{2}{3}}L^{\frac{2}{3}} = 20\lambda$,
$100K^{\frac{1}{3}} \cdot \frac{2}{3} \cdot L^{-\frac{1}{3}} = 10\lambda$,
$20K + 10L - 120,000 = 0$.
To solve the above system, divided the first equation by the second equation, we get $L = 4K$. Thus $K = 2,000$ and $L = 8,000$.