Math 10260 Final Exam Solutions - Spring 2009

1. Since $f(x)$ is a probability density function we have $\int_0^3 f(x)\,dx = 1 - 0.2 = 0.8$.
2. $P(3 < X < 4) = 0.2$
3. The producer surplus (PS) is equal to the area of the rectangle $[0, 8] \times [0, 5]$ minus the area under the supply curve. That is, $PS = 40 - \int_0^8 S(q)dq$. Using the indicated Riemann sum to estimate the integral in this formula we obtain $PS = 40-[2\cdot2+(2.5)\cdot2+3\cdot2+(3.75)\cdot2] = 40 - 22.50 = 17.5$.
4. Equilibrium happens when demand equals suppl. Thus setting $D(q) = S(q)$ gives $\frac{81}{q+3} = q + 3$, or $(q + 3)^2 = 81$, or $q + 3 = \pm 9$, or $q = \pm 9 - 3$. Choosing the positive solution we obtain that $q_e = 6$. Then, the equilibrium price is $p_e = S(6) = 6 + 3 = 9$.
5. At any time t (during the duration of the mortgage) the rate of change of $M(t)$ is equal to rate at which it increases due to interest minus the rate of payments. In symbols, this reads as:
$$\frac{dM}{dt} = 0.05M - 18,000.$$
It is given that $M(0) = 200,000$.
6. We must compare the PV of option (I) with the amount offered in option (II). We have
$$PV = \int_0^{10} 10,000e^{0.1t}e^{-0.05t}dt = 10,000\int_0^{10} e^{0.05t}dt$$
$$= 10,000\frac{e^{0.05t}}{0.05}\Big|_0^{10} = 200,000(e^{0.5} - 1) \approx 129,744.$$
Therefore option (I) is more beneficial. (Note: We could reach the same conclusion by computing FV's.)
7. We have $MPK = \frac{\partial P}{\partial K} = 18 \cdot \frac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}$, which for $K = 1,000$ and $L = 27,000$ is equal to $6(1,000)^{-\frac{2}{3}} \cdot (27,000)^{\frac{2}{3}} = 3 \cdot \frac{900}{100} = 54$.
8. Let $P(x, y)$ denote the profit at the production level (x, y). Then $P(x, y) \approx P(150, 200) + \frac{\partial P}{\partial x}(150, 200)(x - 150) + \frac{\partial P}{\partial y}(150, 200)(y - 200)$
or
$P(x, y) \approx 80,500+100(x-150)+50(y-200)$. Therefore $P(160, 190) \approx 80,500+100(160-150) + 50(190 - 200) = 80,500 + 1,000 - 500 = 81,000$.
9. To find the critical points we must solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Since $\frac{\partial f}{\partial x} = 6x^2 - 6y$ and $\frac{\partial f}{\partial y} = 6y - 6x$, these equations become $6x^2 - 6y = 0$ and $6y - 6x = 0$. The second gives $y = x$, which put into the first gives $6x^2 - 6x = 0$, or $x(x - 1) = 0$, or $x = 0, 1$. For $x = 1$ we get $y = 1$ and for $x = 0$ we obtain $y = 0$. Therefore the critical points are: (1,1) and (0,0).
10. Since $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 3 \cdot 7 - 4^2 = 21 - 16 = 5 > 0$, and $\frac{\partial^2 f}{\partial x^2}(1, 5) = 3 > 0$, we conclude that $f(x, y)$ has a local minimum at (1,5).

11. The error function to be minimized for determining a and b is given by

$$E(a,b) = (a\cdot 2 + b - 3)^2 + (a\cdot 4 + b - 5)^2 + (a\cdot 7 + b - 9)^2.$$

12. $f(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$ achieves its maximum subject to the constraint $4x + y = 8$ at the point (x_0, y_0) where the line $4x + y = 8$ is tangent to a level curve of $f(x,y)$. Looking at the picture given we see that this is the point $A = (1,4)$.

13. The first step of Euler's method gives that $y(0.2) \approx y(0) + y'(0)\cdot\Delta t = 4 + [(0.5)\cdot 4 - 0]\cdot(0.1) = 4.2$. Then the second step gives $y(0.4) \approx 4.2 + [(0.5)(4.2) - 0.1](0.1) = 4.4$.

14. Since the equilibrium solutions are given by $0.01p(5-p) = 0$ or $p = 0$ and $p = 5$, and since $p = 5$ is stable, while $p = 0$ is unstable (draw a few arrows, or apply the stability criterion) we conclude that the solution $p(t)$ corresponding to the initial condition $p(0) = 1$ moves towards $p = 5$ as $t \to \infty$.

15. The picture indicates that $y = 2$ is stable and $y = 0$ is unstable. These must be the equilibrium solutions of the differential equation in $\frac{dy}{dt} = y(2-y)$.

16. We have $f(0) = e^0 = 1, f'(x) = -e^{-x}|_{x=0} = -1 f''(x) = e^{-x}|_{x=0} = 1$ and $f'''(x) = -e^{-x}|_{x=0} = -1$. Therefore $e^{-x} \approx 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3$.

17. It is given that $y(0) = 2$. The differential equation gives $y'(0) = 0^4 + y^2(0) = 0 + 2^2 = 4$. Differentiating the equation $y'(t) = t^4 + y^2(t)$ with respect to t gives $y''(t) = (t^4 + y^2(t))' = 4t^3 + 2y(t)y'(t)$ (Notice that we used the chain rule for differentiating y^2.) Thus $y''(0) = 4\cdot 0^3 + 2y(0)\cdot y'(0) = 2\cdot 2\cdot 4 = 16$. Therefore the 2nd-degree Taylor polynomial approximation of $y(t)$

$$y(t) \approx y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 = 2 + 4t + \frac{16}{2}t^2.$$

18. The PV for the payment at the end of the first year is $180,000e^{-0.07\cdot 1}$. At the end of the second year the payment should be $180,000(1.04)$ and its PV is $180,000(1.04)e^{-0.07\cdot 2}$. Similarly, at the end of the third year the payment should be $180,000(1.04)^2$, and its PV is $180,000(1.04)^2e^{-0.07\cdot 3}$. Continuing like this we find that endowment must be equal to the infinite sum

$$180,000e^{-0.07}[1 + 1.04e^{-0.07} + (1.04e^{-0.07})^2 + \cdots]$$

19. We have $E(X) = 1\cdot(0.3) + 2(0.2) + 3(0.1) + 4(0.3) + 5(0.1) = 2.7$.

20. We have $E(X) = \int_0^1 x\cdot f(x)dx = \int_0^1 x\cdot 3x^2dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$.

21. Let the event H="new-born is female". Then, using the binomial distribution with $X = \#$ of female new-born babies, $n = 4, k = 2$ and $p = 0.49$ we have that the desired probability is equal to

$$P(X=2) = \binom{4}{2}(0.49)^2(0.51)^{4-2} = 0.3747.$$

22. Looking at the given picture we see that $P(5 \le X \le 8) =$ area under the pdf above the interval $[5,8] = 1 - 0.35 - 0.2 = 0.45$.

23. For $f(x) = 1-cx, 0 < x < 2$ be a pdf we must have $1 = \int_0^2 (1-cx)dx = (x - c\frac{x^2}{2})|_0^2 = 2-2c$, or $2c = 1$ or $c = \frac{1}{2}$. In this case we also have $f(x) \geq 0$.

24. Let X="waiting time til the bus arrives". This X is a random variable which is uniformly distributed in the interval [0,20]. Therefore it's pdf is $f(x) = \frac{1}{20}$. Thus

$$E(X) = \int_0^{20} x \cdot f(x)dx = \int_0^{20} x \cdot \frac{1}{20}dx = \frac{1}{20}\frac{x^2}{2}|_0^{20} = 10.$$

25. Since the waiting time X is a random variable with an exponential pdf $f(x) = \lambda e^{-\lambda x}, x \geq 0$, and since we are given that $E(X) = 20$ we must have $\frac{1}{\lambda} = E(X) = 20$, or $\lambda = \frac{1}{20} = 0.05$. Therefore $P(X \geq 30) = \int_{30}^{\infty} 0.05e^{-0.05}dx = -e^{-0.05}|_{30}^{\infty} = e^{-1.5}$.

26. Let X="purchase amount by an ND student". We want $P(25 < X < 40)$. Since $\mu = 30$ and $\sigma = 10$ we have

$$\begin{aligned} P(25 < X < 40) &= P\left(\frac{25-30}{10} < \frac{X-30}{10} < \frac{40-30}{10}\right) \\ &= P(-0.5 < Z < 1) = \Phi(1) - \Phi(-0.5) \\ &= \Phi(1) - [1 - \Phi(0.5)] = 0.8413 - 1 + 0.6915 \\ &= 0.5328 \end{aligned}$$

27. We have $\dfrac{\partial f}{\partial x} = 4y^2e^{4x} + yx^{-1}$ and $\dfrac{\partial^2 f}{\partial y \partial x} = 8ye^{4x} + x^{-1}$.

28. Separating variables we have $\dfrac{dy}{y+1} = 2tdt$. Then, integrating gives $\ln|y+1| = t^2 + c$. Solving for y we get $y = Ce^{t^2} - 1$. Furthermore, using the initial condition we obtain $8 = Ce^0 - 1$, or $C = 9$. Thus, $y = 9e^{t^2} - 1$. Finally, we have $y(1) = 9e - 1$.

29. Coming down the first time the ball travels a distance 6. Then going up and returning the ball covers a distance $2 \cdot (2/3) \cdot 6$. In the second trip up and down the ball covers a distance $2 \cdot (2/3)^2 \cdot 6$ Continuing like this, we see that the ball covers total distance given by the infinite geometric series

$$6+2\left(\frac{2}{3}\right)\cdot 6+2\left(\frac{2}{3}\right)^2\cdot 6+2\left(\frac{2}{3}\right)^3\cdot 6+\cdots = 6+8\left[1+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\cdots\right] = 6+8\cdot\frac{1}{1-(2/3)} = 30.$$

30. Newton **is** famous for his great calculus ideas!