

Math 1260 Exam 3 Solutions – Spring 2009

1.) $\frac{195 + 180}{2073} \approx 0.171$

2.) $f''(2) = P_4''(2) = 0$

3.) Immediately after the injection on the first day, the patient's body contains A mg of the drug. Immediately after the injection on the second day, the amount of the drug is A mg, which was just injected, plus $Ae^{-0.2}$ mg remaining from the amount injected on the first day. Thus, immediately after injection on the second day, the patient's body contains $A + Ae^{-0.2}$ mg of the drug. Immediately after the injection on the third day, the amount of the drug is A mg, which was just injected, plus $Ae^{-0.2}$ mg remaining from the amount injected on the second day, plus $Ae^{-0.4}$ mg remaining from the amount injected on the first day. Thus, immediately after injection on the third day, the patient's body contains $A + Ae^{-0.2} + A(e^{-0.2})^2$ mg of the drug. In a similar way, we see that on the n -th day, the patient's body contains

$$A + Ae^{-0.2} + A(e^{-0.2})^2 + \dots + A(e^{-0.2})^{n-1} = \sum_{k=1}^n A(e^{-0.2})^{k-1} \text{ mg}$$

of the drug.

After a very long time, the patient's body will contain approximately

$$\sum_{k=1}^{\infty} A(e^{-0.2})^{k-1} \text{ mg} = \frac{A}{1 - e^{-0.2}} \text{ mg}$$

of the drug. Since this amount must be kept less than 1000 mg, A must satisfy

$$A < 1000(1 - e^{-0.2}) \approx 181.3$$

4.) Now $P_2(1) = C(1) = 3$, $P_2'(1) = C'(1) > 0$, and $P_2''(1) = C''(1) < 0$. Only $P_2(x) = 3 + 0.7(x - 1) - 0.25(x - 1)^2$ has these properties.

5.) $\sum_{k=0}^{\infty} 5 \left(-\frac{1}{3}\right)^k = \frac{5}{1 - (-\frac{1}{3})} = \frac{15}{4}$

6.) Note that $y_e(t) = 6$ is the largest equilibrium solution of the differential equation, and that it is asymptotically stable, since $g'(6) < 0$. Thus, if $y(t)$ is a solution with $y(0) > 6$, then $\lim_{t \rightarrow \infty} y(t) = 6$. Therefore if the initial population had been greater than 600, we would not have $\lim_{t \rightarrow \infty} y(t) = \infty$.

7.) Let R_1 and R_2 denote the events that the first and second balls drawn are red, respectively, and let G_1 and G_2 denote the events that the first and second balls drawn are green, respectively. Now, $R_2 = (R_1 \cap R_2) \cup (G_1 \cap R_2)$, and the union is disjoint. Thus,

$$\begin{aligned} P(R_2) &= P(R_1 \cap R_2) + P(G_1 \cap R_2) \\ &= P(R_1)P(R_2 | R_1) + P(G_1)P(R_2 | G_1) \\ &= \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{6}{9} \\ &= \frac{3}{5} \end{aligned}$$

8.) In a confined space, we cannot have $\lim_{t \rightarrow \infty} y(t) = \infty$.

9.) Now, $0.15 + p + 0.25 + 2p + 0.3 = 1$, hence $3p = 0.3$. Thus $p = 0.10$.

10.) $E(X) = (20)(0.25) + (30)(0.25) + (50)(0.5) = \frac{75}{2}$

11.) $P_3(t) = y(0) + y'(0)t + \frac{1}{2}y''(0)t^2 + \frac{1}{6}y'''(0)t^3$.

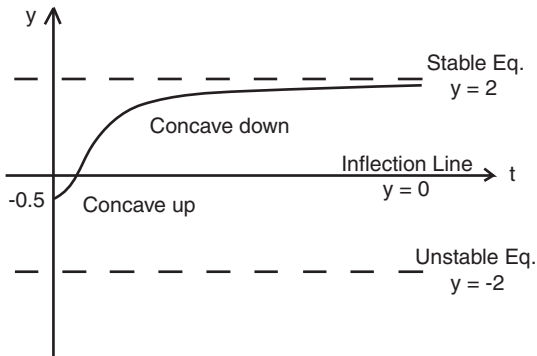
$$\begin{aligned} y(0) &= 0 \\ y'(t) = e^{y(t)} &\Rightarrow y'(0) = 1 \\ y''(t) = e^{y(t)}y'(t) = (y'(t))^2 &\Rightarrow y''(0) = 1 \\ y'''(t) = 2y'(t)y''(t) &\Rightarrow y'''(0) = 2 \end{aligned}$$

Therefore,

$$P_3(t) = t + \frac{1}{2}t^2 + \frac{1}{3}t^3$$

12.) **Part A.** Let $g(y) = 4 - y^2$. Solving $g(y) = 0$, we find the equilibrium solutions $y = \pm 2$. Now $g'(y) = -2y$, so $g'(-2) = 4 > 0$ and $g'(2) = -4 < 0$. Thus the solution $y = -2$ is unstable, and the solution $y = 2$ is asymptotically stable. (One could sketch a graph of g and read this information from the graph.)

Part B.



13.) **Part A.** Let D_k denote the amount of the k -th payment, and let A_k be the present value of D_k at 6% annual interest, compounded continuously. Then $D_1 = \$10,000$, and $A_1 = \$10,000e^{-0.06}$. Also $D_2 = \$10,000(1.04)$, and $A_2 = \$10,000(1.04)(e^{-0.06})^2$. For the end of the third year, we have $D_3 = \$10,000(1.04)^2$ and $A_3 = \$10,000(1.04)^2(e^{-0.06})^3$. Continuing in this way, we see that $D_{10} = \$10,000(1.04)^9$ and

$$A_{10} = \$10,000(1.04)^9(e^{-0.06})^{10} \approx \$7,811.30$$

Part B. $\sum_{k=1}^{\infty} (\$10,000)(e^{-0.06})^k(1.04)^{k-1} = \$10,000e^{-0.06} \frac{1}{1 - 1.04e^{-0.06}} \approx \$457,947.87$

14.) **Part A.** $P(A|M) = 0.081$, $P(A|M^c) = 0.07$, and $P(M) = 0.5$.

Part B. $A = (A \cap M) \cup (A \cap M')$ and the union is disjoint. Thus

$$\begin{aligned} P(A) &= P(M)P(A|M) + P(M')P(A|M') \\ &= (0.5)(0.081) + (0.5)(0.07) \\ &= 0.0755 \end{aligned}$$

Part C.

$$\begin{aligned} P(M|A) &= \frac{P(M \cap A)}{P(A)} \\ &= \frac{P(M)P(A|M)}{P(A)} \\ &= \frac{(0.05)(0.081)}{0.0755} \\ &\approx 0.5364 \end{aligned}$$

15.) Part A. $P_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3$

$$\begin{aligned} f(x) = x^{-1} &\Rightarrow f(1) = 1 \\ f'(x) = -x^{-2} &\Rightarrow f'(1) = -1 \\ f''(x) = 2x^{-3} &\Rightarrow f''(1) = 2 \\ f'''(x) = -6x^{-4} &\Rightarrow f'''(1) = -6 \end{aligned}$$

Thus

$$P_4(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$$

Part B. $\int_0^{0.25} e^{-x^2} dx \approx \int_0^{0.25} P_4(x) dx$, and

$$\begin{aligned} \int_0^{0.25} P_4(x) dx &= \int_0^{0.25} \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4\right) dx \\ &= \left(x - \frac{x^3}{6} + \frac{x^5}{30}\right) \Big|_0^{0.25} \\ &\approx 0.247428 \end{aligned}$$

Thus

$$\int_0^{0.25} e^{-x^2} dx \approx 0.247428$$