SOLUTIONS TO EXAM 2 – MATH 10260, SPRING 2009

1. $(z-3) = 2(x-1) + 5(y+2) = 2x + 5y + 11.$

2. $\sqrt{((2-1)^2 + (1-1)^2 + (1-2)^2} = \sqrt{2}.$

3. We have

$$\frac{\partial f}{\partial x} = y - e^x, \quad \frac{\partial f}{\partial y} = x.$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x = 0, \quad \frac{\partial f}{\partial x} = 0 \Rightarrow y = e^x = 1.$$

Hence $(0,1)$ is the only critical point of the function.

4. The x-slope is -3, the y-slope is -2 and the plane contains the point $(0,0,1)$. Hence its equation is $z - 1 = -3x - 2y + 1$, i.e., $z = -3x - 2y + 1$.

5. We have $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}, \quad \dfrac{\partial^2 f}{\partial x^2} = \dfrac{2(x^2+y^2) - (2x)\cdot(2x)}{(x^2+y^2)^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}.$

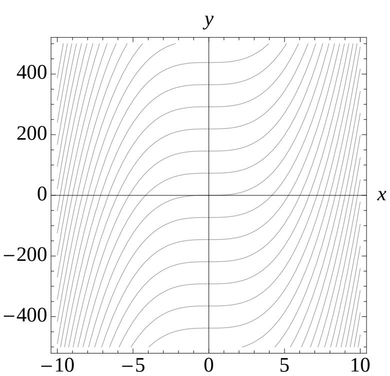
6. The number of vans is approximatively

$$f(25,170) + \frac{\partial f}{\partial x}(25,170)\cdot(0.2) + \frac{\partial f}{\partial y}(25,170)\cdot(-10)$$

$$= 900 + (-180)\cdot(0.2) + (-12)\cdot(-10) = 900 - 36 + 120 = 984.$$

7. $y = 1$ is an equilibrium solution of the differential equation $y' = g(y)$, where $g(y) = y(1-y)$, because $g(1) = 0$ and $g'(1) = 1 - 2 < 0$.

8.



9. Let $g(x,y) = xy$. We need to solve the system

$$g(x,y) = 1, \quad \frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y},$$

i.e.,

$$xy = 1, \quad 4 = \lambda y, \quad 1 = \lambda x \to x = \frac{1}{\lambda}, \quad y = \frac{4}{\lambda}$$

We get $4\lambda^2 xy = 4$, so that $\lambda = \pm 2$. Since $x > 0$ the only critical point is $(\frac{1}{2}, 2)$ and the value of f at this point is 4.

10. The condition $f = const$ implies $x^2 + y^2 - 1$ is constant so the level sets of f are circles with the center at the origin.

11.(a) $\dfrac{\partial p}{\partial x} = 18(3x - 4y + 2) - 4(2x - 3y + 1)$, $\dfrac{\partial p}{\partial y} = 24(4y - 3x - 2) - 6(3y - 2x - 1)$. The critical points satisfy

$$\begin{cases} 30y - 23x &= 16 \\ 13y - 10x &= 7 \end{cases} \implies \boxed{x = -2, \;\; y = -1}.$$

(b) We have

$$\frac{\partial f}{\partial x} = y^2 - 3y + 2, \;\; \frac{\partial f}{\partial y} = (x-5)(2y-3),$$
$$\frac{\partial^2 f}{\partial x^2} = 0, \;\; \frac{\partial f^2}{\partial x \partial y} = 2y - 3.$$

Hence

$$D(5,1) = \underbrace{\frac{\partial^2 f}{\partial x^2}(5,1)}_{=0} \cdot \frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial f^2}{\partial x \partial y}(5,1) \Big)^2 = -1 < 0.$$

Hence $(5,1)$ is a saddle point.

12. (a) We compute the x-slope using the points $(1,3,4)$ and $(2,3,5)$ and we have

$$\boxed{x\text{-slope} = \frac{5-4}{2-1} = 1.}$$

We compute the y-slope using the points $(1,2,3)$ and $(1,3,4)$ and we have

$$\boxed{y\text{-slope} = \frac{4-3}{3-2} = 1.}$$

(b) We have

$$\frac{\partial f}{\partial x} = 2xe^{x^2 - y^4}, \;\; \frac{\partial f}{\partial y} = 4y^3 e^{x^2 - y^4},$$
$$f(1,-1) = 1, \;\; \frac{\partial f}{\partial x}(1,-1) = 2, \;\; \frac{\partial f}{\partial y}(1,-1) = 4$$

The equation of the plane is

$$\boxed{z - 1 = 2(x-1) + 4(y+1)}.$$

13. (a) The error function is

$$E(a,b) = (a + b - 6)^2 + (2a + b - 5)^2 + (3a + b - 3)^2 + (6a + b - 1)^2.$$

(b) We look for the coefficients a and b of the linear function $y = ax + b$ minimizing the error function

$$E(a,b) = (a + b - 5)^2 + (2a + b - 4)^2 + (3a + b - 4)^2.$$

Computing the partial derivatives with respect to a and b gives

$$\frac{\partial E}{\partial a} = 2(a + b - 5) + 4(2a + b - 4) + 6(3a + b - 4)$$

and

$$\frac{\partial E}{\partial b} = 2(a + b - 5) + 2(2a + b - 4) + 2(3a + b - 4).$$

Setting these partials equal to zero and doing the algebra gives the system

$$14a + 6b = 25 \quad \text{and} \quad 6a + 3b = 13.$$

Solving this system we obtain $a = -1/2$ and $b = 16/3$. This is the only critical point of $E(a, b)$. Next we apply the second derivative test. We have

$$\frac{\partial^2 E}{\partial a^2} = 28, \quad \frac{\partial^2 E}{\partial b^2} = 12 \quad \frac{\partial^2 E}{\partial b \partial a} = 0.$$

Since $D = 28 \cdot 12 - 0^2 > 0$ and $\frac{\partial^2 E}{\partial a^2} = 28 > 0$ we conclude that the the minimum of the error function occurs when $a = -1/2$ and $b = 16/3$. Thus, the line fitting the data best is given by $\boxed{y = -\frac{1}{2}x + \frac{16}{3}}$. Letting $x = 0.8$ gives $y = -0.4 + (16/3) \approx 4.93$.

14. (a) $\boxed{c(x, y, z) = 10xy + 5z(2x + y) = 5(2xy + xz + yz)}$.

(b) Let v$g(x, y) = x^2 e^y$. We need to solve the system

$$g(x, y) = 1, \ \ \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \ \ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y},$$

i.e.,

$$x^2 e^y = 1, \ \ 2 = 2\lambda x e^y, \ \ 3 = \lambda x^2 e^y.$$

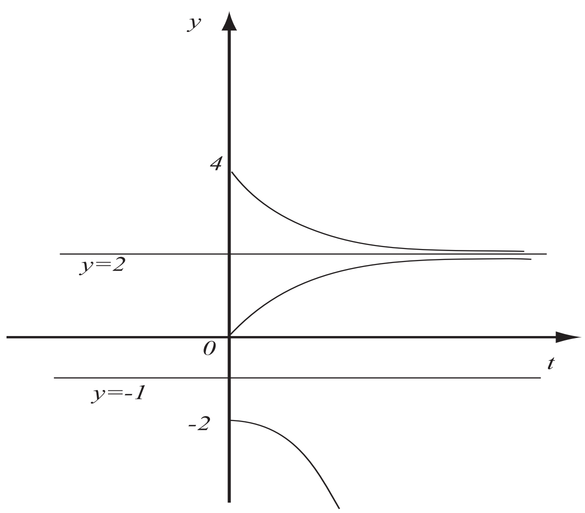
Hence

$$\frac{3}{2} = \frac{\lambda x^2 e^y}{2\lambda x e^y} = \frac{x}{2} \Rightarrow x = 3, \ \ e^y = \frac{1}{x^2} = \frac{1}{9} \Rightarrow y = \ln \frac{1}{9} = -\ln 9.$$

Hence, the only critical point is $(3, -\ln 9)$ and we have

$$\boxed{f(3, -\ln 9) = 6 - 3\ln 9 \approx= -0.591.}$$

15. (a) The equilibrium solutions are $\boxed{y = 2}$ and $\boxed{y = -1}$. The equilibrium solution $y = 2$ is stable.



(b) Letting $g(y) = (2 - y)(y + 1)$

$$y(1/2) \approx y_1 = y_0 + g(y_0)\Delta t = -\frac{1}{2} + (2 + \frac{1}{2})(-\frac{1}{2} + 1) \cdot \frac{1}{2} = \frac{1}{8},$$

$$y(1) \approx y_2 = y_1 + g(y_1)\Delta t = \frac{1}{8} + (2 - \frac{1}{8})(\frac{1}{8} + 1) \cdot \frac{1}{2} = \frac{1}{8} + \frac{135}{128} \approx 1.18.$$

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