Midterm 2 Answers - Bus. Calc. II
October 13, 2009

1. We use the distance formula:

$$\begin{aligned} D((-1,3,2),(2,-2,3)) &= \sqrt{(-1-2)^2+(3-(-2))^2+(2-3)^2} \\ &= \sqrt{35}. \end{aligned}$$

2. This requires an application of the chain rule on $f(x,y) = ye^{x^2+y^2} + 3x^2$:

$$\frac{\partial f}{\partial x} = ye^{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) + 6x = 2xye^{x^2+y^2} + 6x.$$

3. We have to find the critical point of $R(x,y) = 30x + 20y - \frac{1}{2}x^2 - \frac{1}{2}y^2$. We calculate $\frac{\partial R}{\partial x} = 30 - x$, and $\frac{\partial R}{\partial y} = 20 - y$. Setting these two partials equal to zero, we conclude that $x = 30$ and $y = 20$.

4. We first find the equation of the plane passing through $(0,-1,1)$, $(2,0,5)$ and $(0,0,3)$. Using the first point, we must solve for a and b in the equation $z-1 = a(x-0)+b(y-(-1))$. We plug the other two points into this equation to get the two equations $4 = 2a + b$ and $2 = b$. Solving these simultaneous equations, we conclude that the equation of the plane is $z - 1 = x + 2(y+1)$ or simply $z = x + 2y + 3$. The only point provided that satisfies this equation is $(1,1,6)$.

5. We use Lagrange multipliers. Letting $g(x,y,z) = x - 2y + 2z - 13$, we need to solve for (x,y,z) given the equations $4x = \lambda$, $4y = -2\lambda$, $2z = 2\lambda$ and $x - 2y + 2z = 13$. The first two equations give $y = -2x$ and the first and third give $z = 4x$. Substituting these into the last equation gives $x + 4x + 8x = 13$. Thus, $x = 1$, $y = -2$, and $z = 4$. Finally, we compute $f(1,-2,4) = 26$.

6. We have $\frac{\partial f}{\partial x} = x^2 - 3y$, $\frac{\partial f}{\partial y} = 3y^2 - 3x$, and $\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = 6y$, $\frac{\partial^2 f}{\partial x \partial y} = -3$. Therefore $D = (6x)(6y) - (-3)^2 = 36xy - 9$. Since $D(1,1) = 36 - 9 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 6 > 0$, we conclude that f has a local minimum at $(1,1)$.

7. Since we assume the profit function is linear, it is of the form $P(X,Y) = aX + bY + c$. The given data is that $a = 4$, $b = 10$ and $P(40,20) = 400$. Combining these equations, we conclude that $c = 40$. Thus, the profit equation is $P(X,Y) = 4X + 10Y + 40$, and $P(100,50) = 400 + 500 + 40 = 940$.

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}, \quad g(x,y) = 0.$$

In our case this system is $18x = \lambda 3,\ 2y = \lambda 4,\ 3x + 4y - 17 = 0$. Solving the first two equations for λ gives $\lambda = 6x$ and $\lambda = \frac{y}{2}$, or $y = 12x$. Putting this into the equation $3x + 4y = 17$, we obtain $3x + 48x = 17$, which gives $x = \frac{1}{3}$. Then $y = 4$, and we obtain the critical point $(\frac{1}{3}, 4)$.

8. We use Lagrange multipliers with $P(x, y) = 15x + 25y - 0.25(x^2 + y^2)$ subject to the constraint that $g(x, y) = x + y - 100 = 0$. To find the critical points we need to solve the system

$$\frac{\partial P}{\partial x} = \lambda \frac{\partial g}{\partial x}, \quad \frac{\partial P}{\partial y} = \lambda \frac{\partial g}{\partial y}, \quad g(x, y) = 0.$$

which in this case is $15 - 0.5x = \lambda$, $25 - 0.5y = \lambda$, $x + y - 100 = 0$. From the first two equations we get $15 - 0.5x = 25 - 0.5y$ or $y = x + 20$. Substituting that into the third equation gives $2x + 20 = 100$. Thus $x = 40$ and $y = 60$.

9. Write $f(t, y) = t - 2y + 5$, with $n = 2, \Delta t = \frac{1-0}{n} = 0.5, y_0 = 3, t_0 = 0, t_1 = 0.5$, and $t_2 = 1$.

$$y_1 = y_0 + f(t_0, y_0)\Delta t = 3 + (0 - 2(3) + 5)0.5 = 3 - 0.5 = 2.5,$$

$$y_2 = y_1 + f(t_1, y_1)\Delta t = 2.5 + (0.5 - 2(2.5) + 5)0.5 = 2.5 + 0.25 = 2.75.$$

10. We calculate $\frac{\partial P}{\partial K} = 4K^{-3/5}L^{3/5}$, and $\frac{\partial P}{\partial L} = 6K^{2/5}L^{-2/5}$. Evaluating these derivatives at $(32, 243)$, we get that $MPK \approx \frac{27}{2}$ and $MPL \approx \frac{8}{3}$.

11.

(i) We first calculate $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+y}}$ and $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x+y}}$. Evaluating these two partials at $(1, 3)$, we conclude the the slope of the plane in the x-direction is $\frac{1}{4}$, and the slope in the y-direction is also $\frac{1}{4}$. Thus, the equation of the plane is $z - 2 = \frac{1}{4}(x - 1) + \frac{1}{4}(y - 3)$.

(ii) We evaluate the function $z = \frac{1}{4}(x - 1) + \frac{1}{4}(y - 3) + 2$ at the point $(1.2, 3.1)$ to get that $f(x, y) \approx \frac{1}{4}(.2) + \frac{1}{4}(.1) + 2 = 2.075$.

12.

(i) We calculate $\frac{\partial f}{\partial x} = 2xy + 4$ and $\frac{\partial f}{\partial y} = x^2 - 4$. Setting $x^2 - 4 = 0$ we get $x = \pm 2$. Plugging these two values of x into the equation $2xy + 4 = 0$, we get the critical points $(2, -1)$ and $(-2, 1)$.

(ii) We calculate $\frac{\partial^2 f}{\partial x^2} = 2y$, $\frac{\partial^2 f}{\partial y^2} = 0$, and $\frac{\partial^2 f}{\partial y \partial x} = 2x$. Thus, $D = (2y)(0) - (2x)^2 = -4x^2 < 0$. Thus, both points are saddle points.

13.

(i) The revenue formula is given by

$$\begin{aligned} R(x, y) &= x(2y - 3x + 50) + y(x - y + 6) \\ &= -3x^2 - y^2 + 3yx + 6y + 50x \end{aligned}$$

(ii) We calculate $\frac{\partial R}{\partial x} = 3y - 6x + 50$, and $\frac{\partial R}{\partial y} = 3x - 2y + 6$. Setting these two partial equal to zero and solving the corresponding simultaneous equations, we arrive at the critical point $(39\frac{1}{3}, 62)$. To conclude that this is a maximum, we calculate $D = (-6)(-2) - 9 = 3 > 0$, and $\frac{\partial^2 f}{\partial x^2} = -6 < 0$.

14. (i) To find the slope a and the y-intercept b of the least squares line $y = at + b$ through the points $(1, 0), (2, 2)$, and $(3, 3.5)$ we must minimize the "error"

$$E(a, b) = [(a + b) - 0]^2 + [(2a + b) - 2]^2 + [(3a + b) - 3.5]^2.$$

We get

$$\frac{\partial E}{\partial a} = 28a + 12b - 29$$

$$\frac{\partial E}{\partial b} = 12a + 6b - 11.$$

Setting $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$ we obtain $a = \frac{7}{4}$ and $b = -\frac{5}{3}$. Thus, the equation of the least squares line is $y = \frac{7}{4}t - \frac{5}{3}$.

(ii) It $t = 4$, then $y = \frac{7}{4}4 - \frac{5}{3} = \frac{16}{3}$.

15. To maximize $Q(K, L) = 50K^{0.3}L^{0.7}$ subject to the constraint that $P(K, L) = 60K + 20L - 800,000 = 0$, we set

$$\frac{\partial Q}{\partial K} = \lambda \frac{\partial P}{\partial K}, \ \frac{\partial Q}{\partial L} = \lambda \frac{\partial P}{\partial L}.$$

This yields $15K^{-0.7}L^{0.7} = 60\lambda$, $35K^{0.3}L^{-0.3} = 20\lambda$. Dividing the first equation by the second we get $\frac{7}{3}KL^{-1} = \frac{1}{3}$. Thus $L = 7K$. Plugging this into $P(K, L) = 0$ we get $K = 4,000$ and $L = 28,000$.