Math 1260 Exam 1 Solutions – Spring 2009

1.) Taking the integral of both sides of the equation $Q'(x) = 200x^{-1/2}$ we find that $Q(x) = \int 200x^{-1/2}dx = 400x^{1/2} + c$. We can find c by pluging $Q = 500$ and $x = 4$ into the above equation: $500 = 800 + c$ or $c = -300$. Therefore $Q(x) = 400x^{1/2} - 300$, and $Q(25) = 400 \cdot 5 - 300 = 1700$.

2.) The average gas price is equal to $[2.5 + 2.98 + 3.05 + 3.2 + 4.1 + 2.25] \cdot \frac{1}{6} \approx 3.01$ dollars per gallon.

3.) We have $P(100) - P(50) = \int_{50}^{100}(-0.2x + 30)\,dx = (-0.1x^2 + 30x)\Big|_{50}^{100} = [-0.1(100)^2 + 30\cdot 100] - [-0.1(50)^2 + 30 \cdot 50] = 750$.

4.) Applying the future value formula $FV = PVe^{rt}$ in our situation gives $FV = 825e^{(0.02625)10} \approx 1,072.65$ million dollars. This is the amount to be paid by the Taxpayers.

5.) Using midpoints, which is the only choice available here, we have $\int_0^2 f(x)dx \approx [f(0.2) + f(0.6) + f(1) + f(1.4) + f(1.8)] \cdot \Delta x = [7.9 + 8.2 + 8.1 + 7.5 + 6.3](0.4) = 15.2$.

6.) From the shape of the graph of the population $p(t)$ we conclude that $p(t)$ is modeled by a logistic differential equation of the form $\dfrac{dp}{dt} = rp(1 - \dfrac{p}{K})$. Since $p(t)$ goes to 30 as t gets large we conclude that K must be 30. Therefore, we must have $\dfrac{dp}{dt} = rp(1 - \dfrac{p}{30})$, which makes $\dfrac{dp}{dt} = 0.02p(1 - 30p)$ a false statement.

7.) Writing $\dfrac{1}{x^2 - x} = \dfrac{1}{x(x-1)} = \dfrac{1}{x-1} - \dfrac{1}{x}$ gives

$$\int_2^3 \frac{1}{x^2 - x}\,dx = \int_2^3 \left[\frac{1}{x-1} - \frac{1}{x}\right] dx = (\ln|x-1| - \ln|x|)\Big|_2^3 = 2\ln 2 - \ln 3.$$

8.) The consumer surplus is $CS = \int_0^7 D(q)dq - 7 \cdot 10 = 95 - 70 = 25$ (and not 95). The producer surplus is the area of the right triangle with legs 4 and 7. So, $PS = 14$.

9.) Applying the formula for the future value of an income stream $FV = \int_0^T S(t)e^{r(T-t)}dt$ we must have $1,500,000 = \int_0^{35} Se^{0.07(35-t)}dt$. Computing the integral gives the equation $1,500,000 = \frac{S}{0.07}(e^{0.07\cdot 35} - 1)$. Solving it for S gives $S = 1,500,000(0.07)/(e^{0.07\cdot 35} - 1) \approx 9,916.56$ dollars.

10.) The statement "Integral and area are always the same" is false. This is true only for non-negative functions. If a function is negative over an interval then its integral is a negative number and the area between its graph and the x-axis is the opposite of its integral. So, area is always a non-negative number while the integral can be negative. A simple example clarifying this is provided by the function $f(x) = x$, $-1 \le x \le 0$. For this function $\int_{-1}^0 x dx = -1/2$ and the corresponding area is $1/2$.

11.) Writing the given DE in the form $\dfrac{dy}{dt} = (3t^2 - 4)y$ and separating variables we get $\dfrac{dy}{y} = (3t^2 - 4)dt$. Integrating gives $\displaystyle\int \frac{dy}{y} = \int (3t^2 - 4)dt$ or $\ln|y| = t^3 - 4t + c_1$. Exponentiating gives $|y| = e^{c_1}e^{t^3-4t}$ or $y = \pm e^{c_1}e^{t^3-4t}$. Finally, renaming constants gives $y = ce^{t^3-4t}$. Now, using the initial condition $y(2) = 8$ we get $8 = ce^{2^3-4\cdot 2} = c$. Thus, the solution to our initial value problem is $y = 8e^{t^3-4t}$.

12.) A. Letting $u = t$ and $dv = e^{0.1t}dt$ we get $du = dt$ and $v = 10e^{0.1t}$. Now, using integration by parts we have $\int t\, e^{0.1t}dt = uv - \int vdu = 10te^{0.1t} - 10\int e^{0.1t}dt = 10te^{0.1t} - 100e^{0.1t} + c$.

B (i). The rate of change of the mortgage balance M(t) is equal to the rate due to interest minus the rate due to mortgage payments. That is, $\dfrac{dM}{dt} = 0.06M - 18,000$, which is the differential equation modeling M (t).

B (ii). Since the mortgage must be paid by the 30-th year we must have $M(30) = 0$. This condition will allow us to compute the integration constant c involved in the the solution $M(t)$ of the above differential equation. Then computing $M(0)$ gives the initial value of the mortgage.

13.) A. Equating demand and supply gives $100 - 0.05q, = p = 40 + 0.1q$, or $0.15q = 60$, or $q = 60/0.15$. Thus the equilibrium quantity $q_e = 400$. Substituting this value to supply (or demand) equation gives $p = 40 + (0.1)\cdot 400$. Thus the equilibrium price is $p_e = 80$.

B. The consumer surplus is equal to

$$CS = \int_0^8 \frac{90}{q+2}dq - 8\cdot 9 = 90\ln(q+2)\Big|_0^8 - 72 = 90(\ln 10 - \ln 2) - 72 \approx 72.85.$$

The producer surplus is equal to: $PS = 8\cdot 9 - \int_0^8 (q+1)dq = 72 - 0.5(q+1)^2\Big|_0^8 = 72 - 0.5(81-1) = 32$. Also, it can be computed by noticing that it is the area of a right triangle with both legs equal to 8.

14.) A. Making the substitution $u = x^2 - 36$ gives $du = 2xdx$. Also when $x = 6$ the $u = 36 - 36 = 0$ and when $x = 10$ then $u = 100 - 36 = 64$. Thus we have

$$\int_6^{10} x\sqrt{x^2-36}\,dx = \frac{1}{2}\int_0^{64} u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_0^{64} = \frac{1}{3}64^{3/2} = \frac{8^3}{3}.$$

B. Applying the formula for the present value of an income stream $PV = \int_0^T S(t)e^{-rt}dt$ we must have $70,000 = \int_0^{20} Se^{-0.05t}dt$. Computing the integral gives the equation $70,000 = \frac{S}{0.05}(1 - e^{-0.05\cdot 20})$. Solving it for S gives $S = 70,000(0.05)/(1 - e^{-1}) \approx 5,536.92$ dollars per year.

15.) A. Making the substitution $u = x^2 + 1$ gives $du = 2xdx$. Also when $x = 0$ the $u = 1$ and when $x = \infty$ then $u = \infty$ to. Thus we have

$$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \int_1^\infty u^{-2}\,du = \frac{-1}{u}\Big|_1^\infty = \lim_{b\to\infty}\frac{-1}{u}\Big|_1^b = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

B. The present value of the perpetual income stream produced by the highway is equal to:

$$PV = \int_0^\infty S(t)e^{-rt}dt = \int_0^\infty \left[200 - 100e^{-0.01t}\right]e^{-0.04t}dt = 200\int_0^\infty e^{-0.04t}dt - 100\int_0^\infty e^{-0.05t}dt$$
$$= 200\frac{e^{-0.04t}}{-0.04}\Big|_0^\infty - 100\frac{e^{-0.05t}}{-0.05}\Big|_0^\infty = \frac{200}{0.04} - \frac{100}{0.05} = 3,000 \text{ million dollars.}$$