Math 10260 Exam 3 Solutions - Spring 2006

1. There are three equilibrium solutions $p = 0, 2$ and 10. Since $\frac{dp}{dt} = g(p) < 0$ on the interval $0 < p < 2$ and $p > 10$, $p(t)$ is decreasing there. On the other hand, $\frac{dp}{dt} = g(p) > 0$ for $2 < p < 10$, $p(t)$ is increasing there. Therefore, if the initial population $p(0)$ is 3 thousand, $p(t)$ must be increasing. Hence, the species should not become extinct.

2. We set $g(k) = 0.3k^{0.5} - 0.1k = 0$ and solve for k.

   $0 = 0.3k^{0.5} - 0.1k = 0.1k^{0.5}(3 - k^{0.5})$.

   Therefore $k = 0$ or $k^{0.5} = 3$ which implies $k = 9$. Since $g'(9) < 0$, $k = 9$ is a stable solution.

3. $f(x) = 5\ln x, \quad f(1) = 0, \quad f'(x) = \frac{5}{x}, \quad f'(1) = 5,$

   $f''(x) = -\frac{5}{x^2}, \quad f''(1) = -5, \quad f'''(x) = 2\frac{5}{x^3}, \quad f'''(1) = 2 \cdot 5.$

   So the third degree Taylor polynomial is

   $P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$

   $= 0 + 5(x-1) - \frac{5}{2!}(x-1)^2 + \frac{2\cdot 5}{3!}(x-1)^3 = 5(x-1) - \frac{5}{2}(x-1)^2 + \frac{5}{3}(x-1)^3$

4. To get \$10,000 at the end of the first year, the donor will have to deposit $10,000e^{-0.06\cdot 1}$.

   To get another \$10,000 at the end of the second year, the donor will have to deposit additional $10,000e^{-0.06\cdot 2}$. $\cdots$

   Hence to get \$10,000 at the end of every year, the donor will have to deposit at least $10,000e^{-0.06\cdot 1} + 10,000e^{-0.06\cdot 2} + 10,000e^{-0.06\cdot 3} + \cdots$

   $= 10,000e^{-0.06}[1 + e^{-0.06} + e^{-0.06\cdot 2} + \cdots] = \frac{10,000e^{-0.06}}{1 - e^{-0.06}}.$

5. Since near $a = 0$, we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, by replacing x with x^2 we obtain

   $$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{k=0}^{\infty} x^{2k}.$$

6. There are $6^3 = 216$ possible outcomes for rolling three dice and only one possible outcome for getting three 6's. Therefore, probability that you get three 6's is $\frac{1}{216}$.

7. Recall $P(E \cup G) = P(E) + P(G) - P(E \cap G)$. Let E be enjoyed the math class, and G be math major.

   P(a person was a math major **and** enjoyed the class)

   $= P(E \cap G) = P(E) + P(G) - P(E \cup G) = 0.25 + 0.42 - 0.6 = 0.07.$

8. Let G be the first draw is white and E be the second draw is white.

   $P(E \cap G) = P(G) \cdot P(E|G) = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}.$

9. Since the two events are independent, $P(E \cap G) = P(G) \cdot P(E) = 0.6 \times 0.1 = 0.06.$

10. Let X be the value of the ball. The range of X is $\{3, 6\}$.
$P(X = 3) = \frac{2}{6}$ and $P(X = 6) = \frac{4}{6}$. So the expected value $\mu = 3 \cdot \frac{2}{6} + 6 \cdot \frac{4}{6} = 5$.

11. $g(y) = -(y-1)(y-5) = 0$ implies that $y = 1$ and $y = 5$ are two equilibrium solutions. Since $\frac{dy}{dt} = g(y) > 0$ on the interval $1 < y < 5$, $y(t)$ is increasing there. Similarly, since $\frac{dy}{dt} = g(y) < 0$ on the interval $y < 1$ or $y > 5$, $y(t)$ is decreasing there.

For concavity, we need to consider $\frac{d^2y}{dt^2} = g'(y) \cdot g(y)$. Since $g'(3) = 0$, $y = 3$ is an inflection line. Since $g(y)$ is an increasing function on the interval $y < 3$, $g'(y) < 0$ there. Also since $g(y)$ is an decreasing function on the interval $y > 3$, $g'(y) > 0$ there.

By combining information of the signs of both $g(y)$ and $g'(y)$ we have the following:

- for $y < 1$, $g(y) < 0$ and $g'(y) > 0$, $\frac{d^2y}{dt^2} < 0$. $\rightarrow$ $y(t)$ is concave down there.
- for $1 < y < 3$, $g(y) > 0$ and $g'(y) > 0$, $\frac{d^2y}{dt^2} > 0$. $\rightarrow$ $y(t)$ is concave up there.
- for $3 < y < 5$, $g(y) > 0$ and $g'(y) < 0$, $\frac{d^2y}{dt^2} < 0$. $\rightarrow$ $y(t)$ is concave down there.
- for $y > 5$, $g(y) < 0$ and $g'(y) < 0$, $\frac{d^2y}{dt^2} > 0$. $\rightarrow$ $y(t)$ is concave up there.

12. We have $y(0) = 2$. Since $y' = y^2 + 1$, $y'(0) = (2)^2 + 1 = 5$.
By taking derivative both sides and use chain rule, we get $y'' = 2y \cdot y'$,
$y''(0) = 2(2)(5) = 20$.
$y(t) \approx y(0) + y'(0)t + \dfrac{y''(0)}{2}t^2 = 2 + 5t + \dfrac{20}{2}t^2$.
$y(0.1) \approx 2 + 5(0.1) + 10(0.1)^2 = 2.6$.

13. Since $f(x) = e^{-\frac{x^2}{4}} \approx f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \dfrac{f^{(4)}(0)}{4!}x^4$, we can compute $f'(0) \cdots f^{(4)}(0)$ or we can do the following;

Since $e^x \approx 1 + x + \frac{x^2}{2!}$, by replacing x with $-\frac{x^2}{4}$ we obtain $e^{-\frac{x^2}{4}} \approx 1 - \dfrac{x^2}{4} + \dfrac{\frac{x^4}{16}}{2!} = 1 - \dfrac{x^2}{4} + \dfrac{x^4}{32}$

Therefore $\displaystyle\int_0^1 e^{-\frac{x^2}{4}}\ dx \approx \int_0^1 1 - \frac{x^2}{4} + \frac{x^4}{32}\ dx = x - \frac{x^3}{12} + \frac{x^5}{160}\Big|_0^1 = 0.9229.$

14. (a) $P(E|A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. (b) $P(E|B) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$.
(c) $P(E) = P(E \cap A) + P(E \cap B) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{25} = \frac{29}{200}$.
(d) $P(B|E) = \frac{P(E \cap B)}{P(E)} = \frac{\frac{1}{2} \cdot \frac{1}{25}}{\frac{29}{200}} = \frac{4}{29}$.
or $P(B|E) = \dfrac{P(B)P(E|B)}{P(B)P(E|B) + P(B')P(E|B')} = \dfrac{(1/2)(1/25)}{(1/2)(1/25) + (1/2)(1/4)} = \dfrac{4}{29}$.

15. (a) X can take values $\{0, 1, 2, 3, 4\}$.
$P(X = 0) = P(\{HHHH\}) = \frac{1}{16}$;
$P(X = 1) = P(\{THHH, HTHH, HHTH, HHHT\}) = \frac{4}{16}$;
$P(X = 2) = P(\{TTHH, THTH, THHT, HTTH, HTHT, HHTT\}) = \frac{6}{16}$;
$P(X = 3) = P(\{TTTH, THTT, TTHT, HTTT\}) = \frac{4}{16}$;
$P(X = 4) = P(\{TTTT\}) = \frac{1}{16}$.
(b) $P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16}$.
(c) $E(X) = 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = 2$.
(d) $V(X) = 1 \cdot \frac{4}{16} + 4 \cdot \frac{6}{16} + 9 \cdot \frac{4}{16} + 16 \cdot \frac{1}{16} - 4 = 1$. Standard deviation $\sigma(X) = 1$.