Part I: Multiple Choice Questions (5 Points Each)

1. The size of a population of certain species of animals in thousands is modeled by the differential equation

$$\frac{dp}{dt} = g(p)$$

where the graph of $g(p)$ is displayed in Figure 1.

Which of the following statements is **not** true?

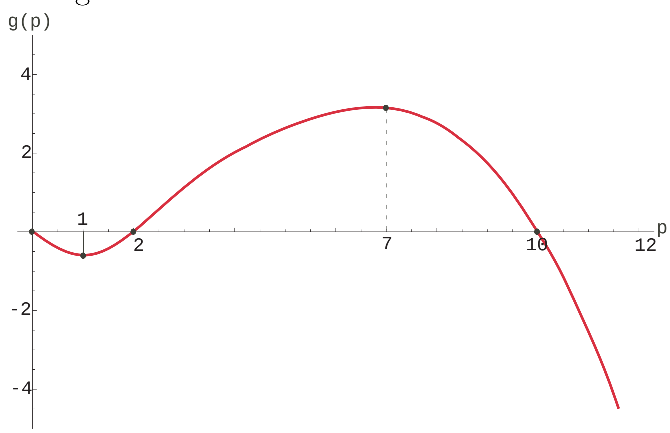


Figure 1.

(a) If the initial population is 3 thousand, this species will become extinct in the long run.

(b) The environmental carrying capacity is 10 thousand.

(c) The threshold level in this model is 2 thousand.

(d) The population will be decreasing if the initial population is more than 10 thousand.

(e) The population will be increasing if the initial population is 6 thousand.

2. Data from a certain source indicates that in the German economy, the investment rate s is 0.3 and the depreciation rate δ is about 0.1. Assume that it follows a Cobb-Douglas production function with $\alpha = 0.5$. Therefore, the corresponding Solow differential equation is

$$\frac{dk}{dt} = 0.3k^{0.5} - 0.1k,$$

where $k(t)$ is capital stock at time t. What would be the asymptotically stable solution?

(a) $k = 10$ **(b)** $k = 3$ **(c)** $k = 9$ **(d)** $k = 0.1$ **(e)** $k = 0$

3. The 3^{rd}-degree Taylor polynomial of $f(x) = 5 \ln x$ about $a = 1$ is

(a) $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

(b) $P_3(x) = 5(x-1) - \frac{5}{2}(x-1)^2 + \frac{5}{3}(x-1)^3$

(c) $P_3(x) = 5(x-1) - \frac{5}{2}(x-1)^2 + \frac{5}{6}(x-1)^3$

(d) $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3$

(e) $P_3(x) = 5x + \frac{5}{2}x^2 + \frac{5}{3}x^3$

4. A donor would like to set a scholarship fund which gives out \$10,000 at the end of every year. Suppose he will deposit a certain amount in an account paying 6% annual interest, compounded continuously. What would be the minimum amount he has to deposit at the beginning to have this scholarship fund last forever?

(a) $\dfrac{e^{-0.06}}{1-e^{-0.06}}$ **(b)** $\dfrac{1}{1-10,000e^{-0.06}}$ **(c)** $\dfrac{1}{1-e^{-0.06}}$

(d) $\dfrac{10,000e^{-0.06}}{1-e^{-0.06}}$ **(e)** infinity.

5. Find the Taylor series for $\dfrac{1}{1-x^2}$ near $a = 0$.

(a) $\sum_{k=0}^{\infty} x^{2k}$ **(b)** $\sum_{k=0}^{\infty} (-1)^k x^{2k}$ **(c)** $\sum_{k=0}^{\infty} (-1)^k x^k$

(d) $\sum_{k=0}^{\infty} x^k$ **(e)** diverges

6. Suppose you roll three dice. What is the probability that you get three 6's?

(a) $\dfrac{1}{6}$ **(b)** $\dfrac{1}{36}$ **(c)** $\dfrac{3}{216}$ **(d)** $\dfrac{3}{6}$ **(e)** $\dfrac{1}{216}$

7. An educational survey queried 100 people about a particular math class. The results were as follows:

- 42 enjoyed the math class
- 25 were math majors.
- 60 were math majors or enjoyed math class (or both).

Find the probability that a person was a math major **and** enjoyed the class.

(a) 0.5 **(b)** 0.07 **(c)** 0.67 **(d)** 0.17 **(e)** 0

8. A bowl contains 9 slips of paper, of which 2 are red, 4 are white, and 3 are blue. Two slips are drawn without replacement from the bowl. The probability that both are *white* is:

(a) $\frac{1}{2}$ **(b)** $\frac{16}{81}$ **(c)** $\frac{1}{6}$ **(d)** $\frac{4}{5}$ **(e)** $\frac{2}{9}$

9. Suppose that there is 60% chance of rain next week in Boston and 10% chance of rain next week in Hawaii. Find the probability that it will rain next week both in Boston and in Hawaii. Assume these events are independent.

(a) 0.06 **(b)** 6 **(c)** 0.7 **(d)** 0.07 **(e)** 0.37

10. A box contains six balls, two of value \$3 each and four of value \$6 each. You pick one of the balls at random from the box. Let X be its value. What is the mean (expected value) μ of X?

(a) $\frac{2}{9}$ **(b)** 30 **(c)** 27 **(d)** 5 **(e)** $\frac{8}{3}$

Part II: Partial Credit Questions (10 Points Each)

11. Consider the differential equation: $\dfrac{dy}{dt} = -(y-1)(y-5)$.

Figure 2 is the graph of $g(y) = -(y-1)(y-5)$.

(a) Find the equilibrium solutions and draw their graphs in the ty-coordinate system below.

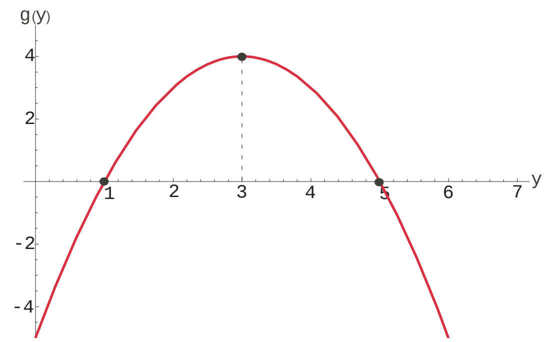
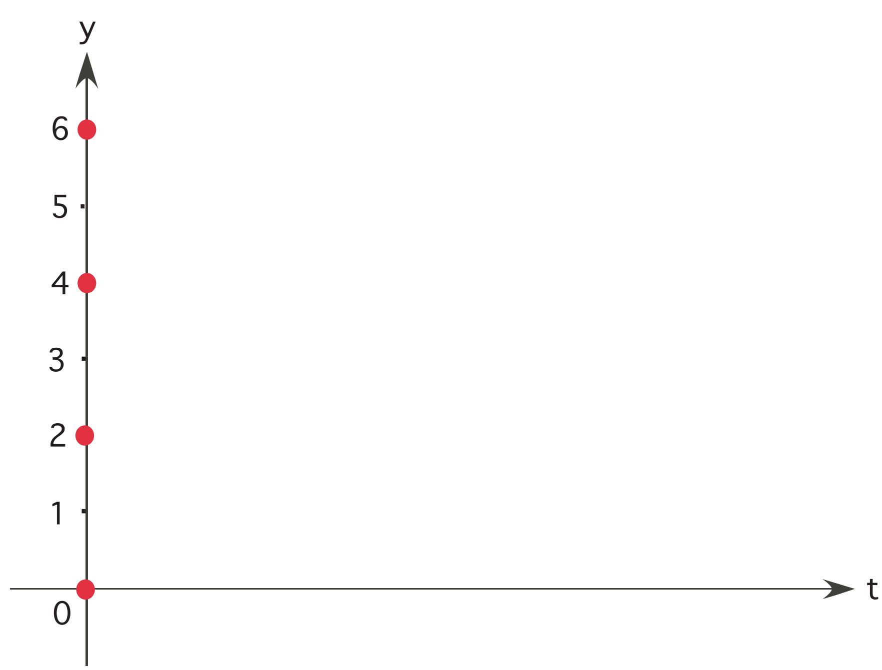


Figure 2

(b) Use the graphical method to sketch the graphs of the solutions to the given initial value problems. In all cases provide explanations for the shape of the graph: increasing/decreasing, concavity, **inflection line**, limiting value.(Make sure to show the graph of **inflection line** and **concavity clearly**.)

$y(0) = 0, \quad y(0) = 2, \quad y(0) = 4, \quad y(0) = 6.$



12. Suppose $y(t)$ is the solution to $y' = y^2 + 1,\ y(0) = 2$. Use the 2^{nd}-degree Taylor polynomial to approximate $y(0.1)$.

13. Estimate the integral $\int_0^1 e^{-x^2/4}\,dx$ using the 4^{th} degree Taylor polynomial of $e^{-x^2/4}$ near $a = 0$.

14. There are two coins in a box. Coin A is a fair coin and the probability of its landing heads is $\frac{1}{2}$. Coin B is weighted so that the probability of heads is $\frac{1}{5}$. Suppose you choose a coin at random and flip it twice. Let $E = \{HH\}$ the event that the coin lands on heads both times. (Hint: A and B are equally likely to be drawn, so: $P(A) = P(B) = \frac{1}{2}$.)

(a) What is the probability that the coin lands heads both times given that it is coin A?

Answer:____________________________

(b) What is the probability that the coin lands heads both times given that it is coin B?

Answer:____________________________

(c) What is the probability that the coin lands heads both times?

Answer:____________________________

(d) What is the probability $P(B|E)$ that it is coin B given that the coin lands heads both times?

Answer:____________________________

15. Let X be the number of heads in four tosses of a fair coin.

(a) Construct the table giving the probability distribution of X.

(b) Compute the probability that $2 \leq X \leq 4$.

Answer:____________________

(c) Compute the expected value of X.

Answer:____________________

(d) Compute the variance and the standard deviation of X.