Math 10260 Exam 2 Solutions - Spring 2006

1. Find the distance between the two points $(1, 2, 3)$ and $(3, 1, 6)$.

   $\sqrt{(1-3)^2 + (2-1)^2 + (3-6)^2} = \boxed{\sqrt{14}}$

2. Suppose that $f(x, y)$ is a function such that

   $$f(3,4) = 10 \qquad \frac{\partial f}{\partial x}(3,4) = 1 \qquad \frac{\partial f}{\partial y}(3,4) = 3$$

   The equation of the tangent plane to the graph of $f(x, y)$ at the point $(3, 4)$ is

   $z - f(3,4) = \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4) \quad \boxed{z = x + 3y - 5}$

3. Suppose that the production function of a certain plant is given by the Cobb-Douglas function

   $$f(K, L) = AK^{\alpha}L^{1-\alpha}$$

   for certain constants A and α. Suppose also that the marginal product of capital (MPK) is 10 units of production for each unit of capital and that the marginal product of labor (MPL) is 20 units of production for each unit of labor. If the plant is currently producing 500 units, use a linear approximation to estimate the production if capital is decreased by one unit while labor is increased by 3 units.

   Production decreases by 10 units due to change in capital and increases by 60 units due to labor - $\boxed{550}$

4. Suppose that a function has partial derivatives

   $$\frac{\partial f}{\partial x} = x^2 - y - 1 \qquad \frac{\partial f}{\partial y} = y - x + 1$$

   Which of the following points is a critical point of $f(x, y)$?

   Of the points listed, $\boxed{(1,0)}$ is the only point that is a zero of both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

5. Compute

   $$\lim_{h \to 0} \frac{ye^{(x+h)y} - ye^{xy}}{h}$$

   This limit is the definition of $\frac{\partial f}{\partial x} ye^{xy}$ and so is $\boxed{y^2e^{yx}}$

6. Which of the following statements is NOT true?

   If at a critical point all the second partial derivatives are equal to zero then there is a minimum at that point. (It could, for example, be a maximum or saddle point.

7. Several contour lines for a certain function $f(x, y)$ are shown below. What is the maximum value of the function $f(x, y)$ subject to the constraint $x + 2y = 10$?

   The line $x + 2y = 10$ is tangent to the contour line for $f(x, y) = \boxed{25}$.

8. The following is a plot of the direction field for which differential equation? $\boxed{y' = 1 - y}$

9. To find the least-squares line of the form $y = ax + b$ for the data

   $$(1, 10) \qquad (2, 4) \qquad (3, 0)$$

   which of the following error functions would you need to minimize? $\boxed{E(a,b) = (a+b-10)^2 + (2a+b-4)^2 + (3a+b)^2}$

10. Which of the following is a graph of the level curves of the function $f(x, y) = x - y^2 + y + 3$. The level curves are parabolas opening to the right.

11. Find the equation of the plane that contains the following three points:

$$(0,1,6) \qquad (1,2,13) \qquad (2,0,8)$$

From the first point, $z = 6 + ax + b(y-1)$. The second and third points give equations $13 = 6 + a + b$ and $8 = 6 + 2a - b$. Thus $a = 3$ and $b = 4$. $\boxed{z = 2 + 3x + 4y}$

12. The function

$$f(x,y) = 3x^2 - 6xy + y^3 - 9y + 1$$

has critical points $(3,3)$, $(-1,-1)$. Use the second derivative test, if possible, to determine whether each critical point is a local maximum, local minimum, or saddle point. If the test is inconclusive, say so. Record your conclusions on the blanks and your reasoning below.

$\frac{\partial^2 f}{\partial x^2} = 6$, $\frac{\partial^2 f}{\partial y^2} = 6y$, and $\frac{\partial^2 f}{\partial x \partial y} = -6$. Thus $D = 36y - 36$. If $y = 3$ then $D > 0$, $\frac{\partial^2 f}{\partial x^2} > 0$ so $(3,3)$ is a local mimimum. If $y = -1$ then $D < 0$ and so $(-1,-1)$ is a saddle point.

13. (a) The Hammes Bookstore sells a book about Rudy in either paperback or hardcover. It has determined that if it charges x dollars for the paperback and y dollars for the hardcover, then it will sell $(100 - 6x + 4y)$ paperbacks and $(120 + 2x - 4y)$ hardcovers in a football weekend.

    i. Write a formula for the weekly revenue as a function of x and y. $R(x,y) = x(100 - 6x + 4y) + y(120 + 2x - 4y)$

    ii. If the books cost the bookstore 4 dollars for each paperback and 9 dollars for each hardcover, write a formula for the cost of the books sold as a function of x and y. $C(x,y) = 4(100-6x+4y)+9(120+2x-4y)$

    iii. If the bookstore people can do calculus, which function would they need to maximize in order to determine the prices to charge to maximize profit? $P(x,y) = R(x,y) - C(x,y)$

    (b) Find the maximum of the following function of x and y. $f(x,y) = -2x^2 + 100x - y^2 + 60y - 2xy - 16$

$\frac{\partial f}{\partial x} = -4x + 100 - 2y$ and $\frac{\partial f}{\partial y} = -2y + 60 - 2x$. The critical point is $x = 20$ and $y = 10$. This is a maximum as $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$.

14. The production of a certain company is modeled by the Cobb-Douglas production function $P(K,L) = 100K^{1/4}L^{3/4}$. Suppose that the budget for the company for capital and labor is \$100,000 and that capital costs are \$25 per unit of capital while labor costs are \$50 per unit of labor. The company seeks to maximize production.

    (a) Express this problem as a constrained optimization problem.
    Maximize $P(K,L) = 100K^{1/4}L^{3/4}$ subject to $25K + 50L = 100000$.

    (b) Use the method of Lagrange multipliers to find the values of capital and labor that result in the maximum production.

The Lagrange multiplier equations are $25K^{-3/4}L^{3/4} = \lambda 25$ and $75K^{1/4}L^{-1/4} = \lambda 50$. We have then $50K^{-3/4}L^{3/4} = 75K^{1/4}L^{-1/4}$ $(= \lambda 50)$. Thus $50L = 75K$ or $L = (3/2)K$. Using this and the constraint gives $100K = 100000$ or $\boxed{K = 1000,\ L = 1500}$

15. (a) The rate of spread of a rumor is often governed by a logistic equation. Suppose that a rumor spreads through your dorm at a rate given by $\frac{dy}{dt} = y\left(1 - \frac{y}{100}\right)$ where $y(t)$ is the number of persons in the dorm who have heard the rumor after t days. Suppose 20 persons have now heard the rumor (i.e., $y(0) = 20$). Use Euler's method with step size $\Delta t = 1/2$ to estimate the number of persons who have heard the rumor after one day.
    $y(1/2) = y(0) + y'(0)(1/2) = 20 + (20)(1 - 20/100)(1/2) = 28$. $y(1) = 28 + 28(1 - 28/100)(1/2) = \boxed{38.08}$.

    (b) For the differential equation $y' = y(y-3)$, determine all the equilibrium solutions and say whether each is stable or unstable.

Obviously $y = 0$ and $y = 3$ are the equilibrium solutions. $g'(y) = 2y - 3$ so $y = 0$ is s stable equilibrium and $y = 3$ is unstable.