Math 10260 Exam 1 - Solutions - Spring 2006

1. Let us set 1980 to stand for $t = 0$. The total amount of oil consumed during 1980 to 2000 = The area under the graph
$\approx [r(5) + r(15)]\,\Delta t = (6 + 8) \cdot 10 = 140.$

2. Using integration by substitution $u = \ln x$. Then $du = \frac{1}{x}dx$,
$\int_1^2 \frac{\ln x}{x} dx = \int_0^{\ln 2} u du.$

3. Total change of profit $= \int_{20}^{30} [-0.5q + 100]\ dq = \frac{-0.5}{2}[q^2 + 100q]\,\Big|_{20}^{30} =$
$\frac{-0.5}{2}(30)^2 + 100(30) - \left[\frac{-0.5}{2}(20)^2 + 100(20)\right] = 875.$

4. CS = [the area under the demand curve from 0 to q_e] – $q_e \cdot p_e$
= the area of the triangle $= \frac{1}{2} \cdot 10 \cdot 2 = 10.$
PS = $q_e \cdot p_e$ – [the area under the supply curve from 0 to q_e] = $3 \cdot 10$ - 18 = 12.

5. Since $g(x) \geq f(x)$ on the interval $[-2, 1]$, then the area of the region between the curves of $f(x)$ and $g(x) = \int_{-2}^{1} [g(x) - f(x)]\, dx = \int_{-2}^{1} [2 - x - x^2]\, dx$

6. Selling price = present value (PV) $= \int_0^8 84000e^{-0.06t}dt = \frac{84000}{-0.06}e^{-0.06t}|_0^8$
$= \frac{84000}{-0.06}\,[e^{-0.06\cdot 8} - 1] =$ \$533,703.

7. The rate of change of $M = \dfrac{dM}{dt}$ = rate due to interest – rate due to payment
$= 0.06M - 9,600.$
Being able to pay off the loan in 20 years means that; $M(20) = 0$.

8. $\dfrac{dp}{dt} = 0.02p - 0.004p^2 = 0.02p(1 - \dfrac{0.004}{0.02}p) = 0.02p(1 - 0.2p) = 0.02p(1 - \dfrac{1}{5}p)$. Therefore $K = 5$ millions.

9. $\text{PV} = \displaystyle\int_0^\infty 65000e^{-0.065t}dt = \lim_{a\to\infty}\int_0^a 65000e^{-0.065t}dt = \lim_{a\to\infty}\frac{65000}{-0.065}e^{-0.065t}|_0^a$
$= \displaystyle\lim_{a\to\infty}\frac{65000}{-0.065}\left[e^{-0.065\cdot a} - 1\right] = 1,000,000$

10. According the graph, the carrying capacity, K, is 10 millions. Therefore the logistic equation should be
$\dfrac{dp}{dt} = rp(1 - \dfrac{p}{10}) = rp(1 - 0.1p).$

11. Comparing Future value
Part a FV= $\displaystyle\int_0^{25} 10,000e^{0.05t}e^{0.05(25-t)}\ dt = \int_0^{25} 10,000e^{0.05t}e^{0.05\cdot 25}e^{-0.05t}\ dt$
$= \displaystyle 10,000e^{1.25}\int_0^{25} dt = 10,000e^{2.25}t\Big|_0^{25} = 10,000e^{1.25}\,[25 - 0] =$ \$872,586.3.
Part b FV = \$850,000.
So we would pick option a.

12. (a) To find the equilibrium quantity q_e and p_e, $D(q) = S(q)$, $\frac{49}{q+3} = q + 3$, $(q + 3)^2 = 49$. Take square root both sides to obtain, $q + 3 = 7$ (we only take the positive quantity.) So $q_e = 4$. Then plug q_e in the $S(q)$, we get $p_e = 4 + 3 = 7$.
(b) $\text{CS} = \displaystyle\int_0^{q_e} D(q)dq - q_e \cdot p_e = \int_0^4 \frac{49}{q+3}dq - 4 \cdot 7$
$= 49 \ln |q + 3|\Big|_0^4 - 28 = 49(\ln 7 - \ln 3) - 28 = 49 \ln \frac{7}{3} - 28 = 13.52.$

(c) PS $= q_e \cdot p_e - \int_0^{q_e} S(q)dq = 4 \cdot 7 - \int_0^4 q + 3\ dq = 28 - (\frac{q^2}{2} + 3q)\Big|_0^4$
$= 28 - \left[\frac{4^2}{2} + 3(4) - 0\right] = 8.$

13. (a) $\frac{dy}{dx} = 3x^3(y-1)$, $\frac{1}{y-1}dy = 3x^2dx$, Then integrate both sides.
$\int \frac{1}{y-1}dy = \int 3x^2dx$, $\ln|y-1| = x^3 + c$. Then exponentiate both sides to obtain,
$e^{\ln|y-1|} = |y-1| = e^{x^3+c} = e^c e^{x^3}$,
Hence, $|y-1| = e^c e^{x^3}$, $y - 1 = \pm e^c e^{x^3} = \pm Ce^{x^3}$ where $C = \pm e^c$. Hence, $y = Ce^{x^3} + 1$.
Use the initial condition, $y(0) = 2$, we get $2 = Ce^0 + 1$, $C = 1$.
So $y = e^{x^3} + 1$.
(b) If $y(x) = x^4 + 2$, then $xy' - 4y = x(4x^3) - 4(x^4 + 2) = 4x^4 - 4x^4 - 8 = 8$ which is the same as the differential equation in part b.

14. In order to find the rate of withdraw S per year, we may compare the present value of both cases. i.e. the present value of $1,000,000 now has to be equal to the present value of the income stream with a steady rate S. Hence, we have
$1{,}000{,}000 = \int_0^{50} Se^{-0.05t}dt = S\int_0^{50} e^{-0.05t}dt = \frac{S}{-0.05}e^{-0.05t}\Big|_0^{50}.$
Hence, $1{,}000{,}000 = \frac{S}{-0.05}\left[e^{-0.05\cdot 50} - 1\right]$. Therefore $S = \frac{0.05 \cdot 1,000,000}{1 - e^{-2.5}} = \$54{,}471.28$.

15. (a) The definite integral of $\int xe^{-x^2}\ dx$ can be computed by using $u-$substitiution.
$u = -x^2$, $du = -2x\ dx$, Hence $\int xe^{-x^2}\ dx = -\frac{1}{2}\int e^u\ du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c.$
We learned that $\int_0^{\infty} xe^{-x^2}\ dx = \lim_{a\to\infty}\int_0^a xe^{-x^2}\ dx.$
Since the definite integral of $\int xe^{-x^2}\ dx = -\frac{1}{2}e^{-x^2} + c$, we have
$\lim_{a\to\infty}\int_0^a xe^{-x^2}\ dx = \lim_{a\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^a = -\frac{1}{2}\lim_{a\to\infty}\left[e^{-a^2} - 1\right] = \frac{1}{2}.$
(b) The PV of a perpetual income stream is equal to:
$\int_0^{\infty} S(t)e^{-rt}\ dt = \int_0^{\infty}(t^2e^{-0.1t} + 100)e^{-0.05t}\ dt.$