Practice A – Math 1250 Final Exam Solutions

1. Isaac Newton is famous for his calculus ideas and **not** for the delicious fig newtons!

2. The slope is $m = \dfrac{\Delta q}{\Delta p} = \dfrac{50-90}{7-5} = \dfrac{-40}{2} = -20$, and the equation is $q - 90 = -20(p-5)$, or $q = -20p + 190$.

3. The revenue function is $R = q \cdot p = q(-0.1q + 40) = -0.1q^2 + 40q$, while the cost function is $C = 5q$. Therefore, the profit function is $P = R - C = -0.1q^2 + 40q - 5q = -0.1q^2 + 35q$.

4. The marginal profit is the derivative of the profit function. Using the quotient rule we get $P'(x) = \dfrac{50(x+4) - 50x(1+0)}{(x+4)^2} - 2 = \dfrac{200}{(x+4)^2} - 2$. Thus $P'(6) = \dfrac{200}{(6+4)^2} - 2 = \dfrac{200}{100} - 2 = 0$.

5. The instantaneous rate of change of the tea temperature at time t is equal to $H'(t) = 400e^{-0.1t} \cdot (-0.1) = -40e^{-0.1t}$. Therefore at $t = 10$ we have $H'(10) = -40e^{-0.1 \cdot 10} = -40e^{-1}$.

6. Since by the definition of the derivative of the function $\ln x$ we have $\dfrac{d}{dx}\ln x = \lim\limits_{h \to 0} \dfrac{\ln(x+h) - \ln x}{h}$, and since $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ the given limit is equal to the derivative of $\ln x$ at $x = 5$, which is equal to $1/5 = 0.2$.

7. We have that $f'(6)$ is the slope of the tangent line to the graph of $f(x)$ at $x = 6$. Reading the information displayed in the given figure we have $f'(6) = \dfrac{\Delta y}{\Delta x} = \dfrac{-8}{4} = -2$.

8. Reading the information displayed in the given figure we have $f'(1) = \dfrac{\Delta y}{\Delta x} = \dfrac{6-2}{2-0} = 2$, and $f(1.2) \approx f(1) + f'(1)(1.2 - 1) = 4 + 2(0.2) = 4.4$.

9. Reading the given figure we get $g'(6) = \dfrac{\Delta y}{\Delta x} = \dfrac{4}{2} = 2$. Then, applying the chain rule for exponentials we have $[e^{g(x)}]' = e^{g(x)} \cdot g'(x)$. Therefore, the derivative of the function $e^{g(x)}$ at $x = 6$ is equal to $e^{g(6)} \cdot g'(6) = e^5 \cdot 2 = 2e^5$.

10. We have $f'(x) = 4x^3 - 6x^2 + 4x$ and $f''(x) = 12x^2 - 12x + 4$. Thus, $f''(1) = 12 - 12 + 4 = 4$.

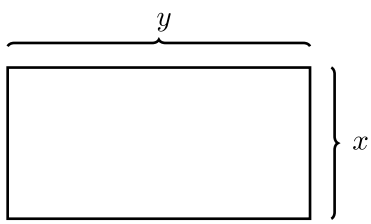
11. At $x = 2$ we have a vertical tangent line, at $x = 5, 13$ $f(x)$ is discontinuous, and at $x = 9$ there is a sharp corner. At all other points the graph of $f(x)$ looks smooth. Thus, $f(x)$ is **not** differentiable at $x = 2, 5, 9, 13$.

12. Applying the quotient rule we get $\left[\dfrac{f(t)}{t^3 - t + 1}\right]' = \dfrac{(t^3 - t + 1) \cdot f'(t) - (3t^2 - 1) \cdot f(t)}{(t^3 - t + 1)^2}$. Therefore, the instantaneous rate of change of the given quantity at $t = 1$ is equal to $\dfrac{(1) \cdot f'(1) - (2) \cdot f(1)}{(1)^2} = 3 - 4 = -1$.

13. Property (i) is satisfied by all graphs except (IV). Property (ii) is satisfied by all graphs except (IV). Property (iii) is satisfied only by graph (I), which is also satisfying Property (iv). Thus, Graph (I) is the only possible graph for the given function.

14. Taking derivative on both sides, we get $e^y + xe^y\frac{dy}{dx} + \frac{dy}{dx}e^x + ye^x = 0$. Solving for $\frac{dy}{dx}$, we get $\frac{dy}{dx}(xe^y + e^x) = -(e^y + ye^x)$, which implies $\frac{dy}{dx} = -\frac{e^y + ye^x}{xe^y + e^x}$.

15. We have $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$, which gives $\frac{dA}{dt} = 2\pi \cdot 50 \cdot 8 = 800\pi$.

16. 

We have $2x + 2y = 20$, which implies that $y = 10 - x$. We want to maxmize the function $A(x) = x \cdot y = x \cdot (10 - x) = -x^2 + 10x$, for $0 \le x \le 10$. Taking derivative, we get $A'(x) = -2x + 10$. Let $A'(x) = 0$, we get $x = 5$, which implies that the critical point is at $x = 5$. Evaluating $A(x)$ at $0, 5, 10$, we get

x	0	5	10
$A(x)$	0	25	0

Therefore the largest area that she can enclose is 25.

17. Since $f''(x) = 20x^3 + 60x^2 = 20x^2(x + 3) = 0$ we have that the sign of $f''(x)$ changes only at $x = -3$. Thus $x = -3$ is the only inflection point.

18. We have $C'(x) = 10 - \frac{90}{(x+1)^2}$. Solving $C'(x) = 0$ or $10 - \frac{90}{(x+1)^2} = 0$ we find $(x+1)^2 = 9$, or $x = \pm 3 - 1$. Thus $x = 2$ is the only critical point inside the interval $(0, \infty)$. Since $C(0) = 170$, $C(2) = 130$ and $\lim_{x\to\infty} C(x) = \infty$ we conclude that $C(x)$ attains a global minimum at $x = 2$. This minimum is $C(2) = 130$.

19. Let us denote by x the distance from point A to B. Then, using the Pythagorean theorem we see that the length of the line segment FB through the water is equal to $\sqrt{x^2 + 8^2}$. Therefore, the time needed to be traveled by the dog in the water is equal to $\frac{\sqrt{x^2 + 64}}{10}$ minutes. The line segment on the beach BD has length $(80 - x)$. So it needs $\frac{80 - x}{100}$ minutes. Therefore, the time needed to be traveled by the dog to reach the frisbee is equal $f(x) = \frac{\sqrt{x^2 + 64}}{10} + \frac{80 - x}{100}$, and this is the function that needs to be minimized for determining the point B.

20. We have $\int_{-2}^{4} f(x)dx = -2 + 2 + 4 = 4$, and not 8.

21. Using integration by parts with $u = \ln x$ and $dv = x^8 dx$ we find $du = \frac{1}{x}dx$ and $v = \frac{1}{9}x^9$. Thus

$$\int \ln x \cdot x^8 dx = \frac{1}{9}x^9 \ln x - \int \frac{1}{9}x^9 \frac{1}{x} dx = \frac{1}{9}x^9 \ln x - \frac{1}{81}x^9 + c.$$

22. We have $\int_0^2 f(x)dx \approx [-5 - 3 + 4 + 7](0.5) = 3(0.5) = 1.5$

23. Using the substitution $u = x^4 + 4x$ we find $du = 4(x^3 + 1)dx$ or $(x^3 + 1)dx = \frac{1}{4}du$. Also, when $x = 0$ then we have $u = 0^4 + 4 \cdot 0 = 0$ and when $x = 1$ then $u = 1^4 + 4 \cdot 1 = 5$. Thus,

$$\int_0^1 (x+1)e^{x^4+4x}dx = \frac{1}{4}\int_0^5 e^u du = \frac{1}{4}e^u\Big|_0^5 = \frac{1}{4}(e^5 - 1).$$

24. Since on the interval $[1,5]$ we have $f(x) \geq g(x)$ and the interval $[5,7]$ we have $f(x) \leq g(x)$ the area between the curves $y = f(x)$ and $y = g(x)$ is given be the expression $\int_1^5 [f(x) - g(x)]dx + \int_5^7 [g(x) - f(x)]dx$.

25. From the Fundamental Theorem of Calculus we have the definite integral of the marginal profit between two production levels gives the total change of the profit. Thus, in this case the total change in profit is equal to

$$\int_{40}^{50} [-0.4x + 20]dx = [-0.2x^2 + 20x]\Big|_{40}^{50} = [-0.2(2500) + 20(50)] - [-0.2(1600) + 20(40)] = 20.$$

26. From the Fundamental Theorem of Calculus we have that the total revenue is equal to the area under the marginal revenue. Thus, $\int_0^1 R'(x)dx \approx [10+30+50+40+20] \cdot (0.2) = 150(0.2) = 30$.

27. At any time t the amount of money in the account is equal to $8000e^{0.04t}$. Thus, the average amount of money in this account during the first 5 years is equal to $\frac{1}{5-0} \int_0^5 8000e^{0.04t}dt$.

28. Since $F(8) = \int_0^8 f(x)dx = 16$ is the area under the graph of $f(x)$ from 0 to 8 and since $f(x) < 5$ in most part of this interval we have that $F(8)$ can **not** be equal to 40. All the other statements are true by the the Fundamental Theorem of Calculus and the information displayed in the given figure.

29. The consumer surplus is equal to

$$CS = \int_0^3 \frac{16}{q+1}dq - 3 \cdot 4 = 16 \ln(q+1)\Big|_0^3 - 12 = 16 \ln 4 - 12 \approx 10.18.$$

30. To find the present value of the given IRA we use the formula $PV = \int_0^T S(t)e^{-rt}dt$ with $S(t) = 2,500e^{0.2t}$, $T = 20$, and $r = 0.1$. Thus, we get

$$PV = \int_0^{20} 2500e^{0.2t}e^{-0.1t}dt = 2500 \int_0^{20} e^{0.1t}dt = 2500 \frac{e^{01.t}}{0.1}\Big|_0^{20} = 25000(e^2 - 1).$$