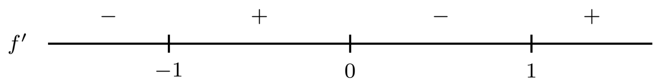
Practice A – Math 10250 Exam 3 Solutions

1.) Use implicit differentiation. $2x + 2y\frac{dy}{dx} = xe^y\frac{dy}{dx} + e^y$. Hence $\frac{dy}{dx}[2y - xe^y] = e^y - 2x$, so $\frac{dy}{dx} = \frac{e^y - 2x}{2y - xe^y}$.

2.) Note that $f'(x) = e^{x-2}(x)(x-1)(x+1)$ and that e^{x-2} is always positive. We get



So there is a local minimum at $x = -1$ and at $x = 1$, and a local maximum at $x = 0$.

3.) The instructions say that the domain of $f(x)$ is all x. The critical points are the points of the domain of $f(x)$ where $f'(x) = 0$ or $f'(x)$ is undefined. From the graph this is clearly $x = 1, 2, 3$.

4.) The graph is concave up on intervals where $f'(x)$ is increasing, so it is concave up on $(-\infty, 0)$ and on $(2, \infty)$.

5.) Since $f(x)$ is differentiable, $f'(x)$ exists for all x and $f(x)$ is continuous; in particular, no asymptotes. The conditions say that $f(x)$ is increasing from $-\infty$ to 1, then decreasing from 1 to 2, then increasing from 2 to 3, then decreasing from 3 to ∞. So there is a local max at $x = 1$ and at $x = 3$, but the one at $x = 3$ is higher than the one at $x = 1$ (since $f(1) < f(3)$). Also there is a local min at $x = 2$. From the shape just described, the highest point is at $x = 3$, so there's a global max there. Since $f(x)$ is decreasing from $x = 1$ to $x = 2$, $f(2) < f(1)$. But since $\lim_{x\to\infty} f(x) = -\infty$, it can't be true that the local min at $x = 2$ is actually a global min.

6.) Since $f'(1) = 0$ and $f''(1) > 0$, there is a local min at $x = 1$. Since $f'(3) = 0$ and $f''(3) < 0$, there is a local max at $x = 3$. Since $f''(x)$ changes from positive to negative at $x = 2$, there is an inflection point there. Since there are no critical points between $x = 1$ and $x = 3$, and $f'(2) > 0$, $f(x)$ must be increasing from $x = 1$ to $x = 3$. Since $f(x)$ is increasing from $x = 1$ to $x = 3$ and $f(1) = 2$, we must have $f(2) > f(1) = 2$ so it can't be true that $f(2) < 2$.

7.) The volume of a rectangular box, whose square base has a side of length x and whose height is y, is x^2y. Since the volume is 100, we get $y = \frac{100}{x^2}$. Each side of the box has area $xy = x \cdot \frac{100}{x^2} = \frac{100}{x}$, but since there are four sides the contribution from the sides is $\frac{400}{x}$. The base and top each have area x^2. So the total area is $A(x) = 2x^2 + \frac{400}{x}$.

8.) We want to find an indefinite integral for $x^{1/2} + \frac{1}{x} + e^{4x}$. This is $\frac{2}{3}x^{3/2} + \ln x + \frac{1}{4}e^{4x} + C$. Notice that since $x > 0$, we don't need to write $\ln|x|$.

9.) If $u = x^2 + 1$ then $du = 2xdx$, so $xdx = \frac{1}{2}du$. We also have $x^2 = u - 1$. Thus our integral becomes

$$\int x^3 e^{x^2+1}dx = \int x^2 e^{x^2+1} \cdot xdx = \frac{1}{2}\int (u-1)e^u du.$$

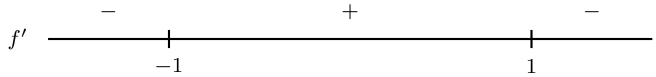
10.) $f'(x) = 3x^2 - 12$, so the critical points are $x = 2, -2$. Notice that -2 is not in the interval! So we evaluate $f(x)$ at $x = -1, 2, 3$: $f(-1) = 15, f(2) = -12, f(3) = -5$. Thus the maximum is 15.

11.) (a) $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$. (b) Both are decreasing. (c) From (a) we have $-48\pi = 4\pi r^2(-1/12)$, so $r^2 = 144$ and $r = 12$.

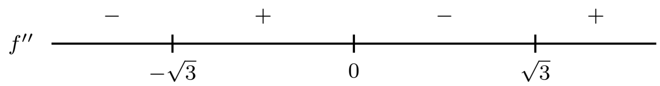
12.) (a) Since x^2 has degree 2 and x has degree 1, $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, so the x-axis is a horizontal asymptote. Since $x^2 + 1$ is never zero, there is no vertical asymptote.

(b) Both $f(x)$ and $f'(x)$ are defined for all x. Setting $f'(x) = 0$ we get that the only critical points are $x = -1, 1$.

(c) We have



so $f(x)$ is decreasing on $(-\infty,-1)$ and on $(1,\infty)$, and increasing on $(-1,1)$.
(d) $f''(x)=0$ for $x=-\sqrt{3},0,\sqrt{3}$ and is defined everywhere. We have



so $f(x)$ is concave up on $(-\sqrt{3},0)$ and on $(\sqrt{3},\infty)$, while it's concave down on $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$.
(e) From (c) we get local minimum at $x=-1$ and a local maximum at $x=1$.

13.) (a) We have $\frac{dy}{dx}=x^{-1/2}-xe^{x^2}$, so integrating we get $y=2x^{1/2}-\int xe^{x^2}dx+C$. To find the remaining integral, we use the substitution $u=x^2$, so $du=2xdx$ and $\frac{1}{2}du=xdx$. Then

$$\int xe^{x^2}dx=\frac{1}{2}\int e^u du=\frac{1}{2}e^u+C=\frac{1}{2}e^{x^2}+C.$$

Putting them together, we get $y=2x^{1/2}-\frac{1}{2}e^{x^2}+C$. But we have $y(1)=4$, so

$$4=2(1)^{1/2}-\frac{1}{2}e^{1^2}+C=2-\frac{1}{2}e+C,$$

i.e. $C=2+\frac{1}{2}e$, and $y=2x^{1/2}-\frac{1}{2}e^{x^2}+2+\frac{1}{2}e$.

(b) Use the substitution $u=x^2+1$, $du=2xdx$, so $\frac{1}{2}du=xdx$. Then we get

$$\int\frac{x\ln(x^2+1)}{x^2+1}dx=\frac{1}{2}\int\frac{\ln(u)}{u}du.$$

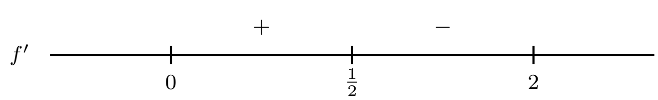
Now we make a new substitution $v=\ln(u)$, so $dv=\frac{1}{u}du$. Then the latter integral becomes

$$\frac{1}{2}\int vdv=\frac{1}{4}v^2+C=\frac{1}{4}(\ln(u))^2+C=\frac{1}{4}(\ln(x^2+1))^2+C.$$

14.)
(a) $0\le x\le 2$.
(b) $R=pq=qe^{8-2q^2}$.
(c) We have

$$\frac{dR}{dq}=q[e^{8-2q^2}(-4q)]+e^{8-2q^2}(1)=e^{8-2q^2}(1-4q^2)=e^{8-2q^2}(1-2q)(1+2q).$$

(d) No power of e is ever zero, so the critical points are when $1-4q^2=0$, i.e. $q=\frac{1}{2}$ and $q=-\frac{1}{2}$. However, $-\frac{1}{2}$ is outside the domain, so the only critical point is $q=\frac{1}{2}$.
(e) We have



so R is maximized at $q=\frac{1}{2}$.