Name ________________________________   Date ____________________

Math 10250 Activity 35: Computing Definite Integrals, Areas, Averages, & Numerics (5.6-6.1)

GOAL: Apply the method of integration by parts with the Fundamental Theorem of Calculus to compute Integrals, Areas, Averages. Also, estimate integrals via numerical methods. Finally, compute Consumer and Producer Surplus for given demand and supply curves.

▸ Integration by parts in definite integrals $\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$

Example 1

(a) $\displaystyle\int_0^1 te^{t/2}\,dt = 2te^t\Big|_0^1 - 2\int_0^1 e^{\frac{t}{2}}dt$

$u = t$
$du = dt$
$e^{\frac{t}{2}}dt = dv$
$2e^{\frac{t}{2}} = v$

$\displaystyle = \left[2te^{\frac{t}{2}} - 2(2e^{\frac{t}{2}})\right]\Big|_0^1$

$= 2\cdot e^{1/2} - 4e^{1/2} - (0-4)$

$= -2e^{1/2} + 4$

(b) $\displaystyle\int_1^2 x\ln x\,dx = \ln x\cdot\frac{x^2}{2}\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{dx}{x} = \ln x\cdot\frac{x^2}{2}\Big|_1^2 - \frac{1}{2}\int_1^2 xdx$

$u = \ln x$
$du = \frac{dx}{x}$
$xdx = dv$
$v = \frac{x^2}{2}$

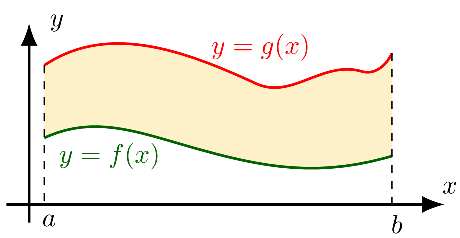
$\displaystyle = \left[\frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2}\right]\Big|_1^2 = \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]\Big|_1^2$

$\displaystyle = \frac{2^2}{2}\ln 2 - \frac{2^2}{4} - \left(\frac{1^2}{2}\ln 1 - \frac{1^2}{4}\right)$

$\displaystyle = 2\ln 2 - 1 + \frac{1}{4} = 2\ln 2 - \frac{3}{4}$

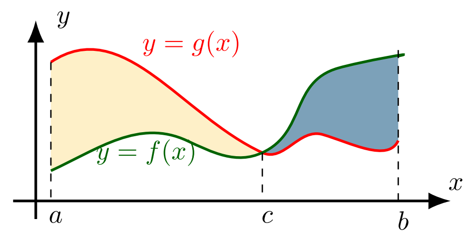
▸ The area between two curves

Consider the following region:



Area between f and $g = \int_a^b$ *up − down*.

Now consider:

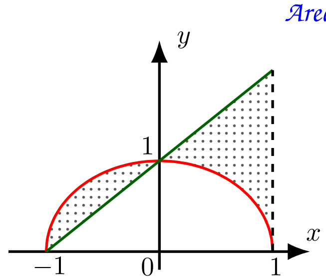


Area between f and $g = \int_a^c [g(x) - f(x)]\,dx + \int_c^b [f(x) - g(x)]\,dx$.

Example 2 Find the intersection points of $f(x) = 1 - x^2$ and $g(x) = x + 1$. Then find the area between the graphs over the interval $-1 \le x \le 1$.

Find the intersection point at x

$1 - x^2 = x + 1$
$\iff x^2 + x = 0$
$\iff x(x+1) = 0$
$\iff x = -1, 0$



$\displaystyle \textit{Area} = \int_{-1}^0 [(1-x^2) - (x+1)]dx + \int_0^1 [(x+1) - (1-x^2)]dx$

$\displaystyle = \int_{-1}^0 [-x^2 - x]dx + \int_0^1 [x^2 + x]dx$

$\displaystyle = -\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^0 + \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$

$\displaystyle = \left(-\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{2}\right) = \boxed{1}$

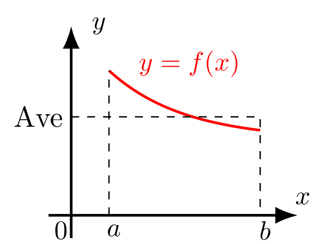
▸ Average values of continuous quantities

Q1: What is the average value of $3, 5$ and 7?

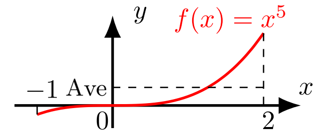
A1: $\frac{3+5+7}{3} = 5$

Q2: What is the average value of $f(x)$ on $[a, b]$?

A2: $\boxed{\text{Average value of } f \text{ over } [a,b] = \frac{\int_a^b f(x)dx}{b-a}}$



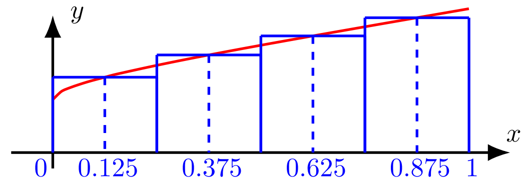
Example 3 Find the average value of $f(x) = x^5$ over the interval $[-1, 2]$.



$$\frac{\int_a^b f(x)dx}{b-a} = \frac{\int_{-1}^2 x^5 dx}{2-(-1)} = \frac{1}{3}\cdot\frac{x^6}{6}\Big|_{-1}^{2} = \frac{1}{18}[2^6 - (-1)^6] = \frac{63}{18} = \boxed{\frac{7}{2}}$$

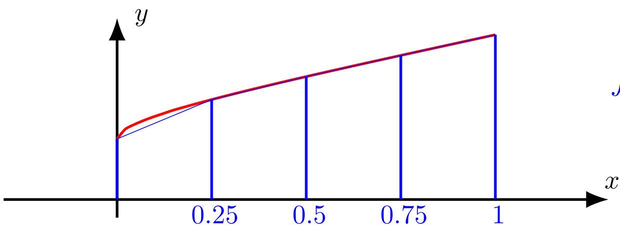
Example 4 Estimate $\int_0^1 e^{\sqrt{x}}\, dx$

(a) using the mid-point with $n = 4$.



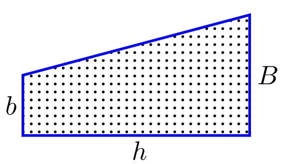
$$\int_0^1 e^{\sqrt{x}}\, dx \approx \left[e^{\sqrt{0.125}} + e^{\sqrt{0.375}} + e^{\sqrt{0.625}} + e^{\sqrt{0.875}}\right]\cdot 0.25$$

(b) using trapezoidal rule with $n = 4$. (See description below.)

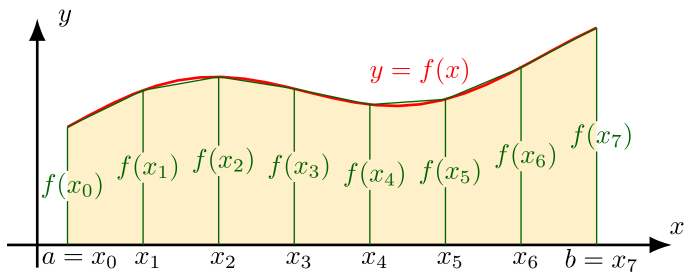


$$\int_0^1 e^{\sqrt{x}}dx \approx \left[e^{\sqrt{0}} + e^{\sqrt{0.25}}\right]\cdot\frac{0.25}{2} + \left[e^{\sqrt{0.25}} + e^{\sqrt{0.5}}\right]\cdot\frac{0.25}{2}$$
$$+ \left[e^{\sqrt{0.5}} + e^{\sqrt{0.75}}\right]\cdot\frac{0.25}{2} + \left[e^{\sqrt{0.75}} + e^{\sqrt{1}}\right]\cdot\frac{0.25}{2}$$
$$= \left[e^{\sqrt{0}} + 2e^{\sqrt{0.25}} + 2e^{\sqrt{0.5}} + 2e^{\sqrt{0.75}} + e^{\sqrt{1}}\right]\cdot\frac{0.25}{2}$$

▶ **The trapezoidal rule:** To estimate $\int_a^b f(x)\, dx$, we can use trapezoids instead of rectangles. Recall that the area of a trapezoid $= \frac{1}{2}$(sum of the two parallel sides)·(height).



$$A = \frac{b+B}{2}\cdot h$$

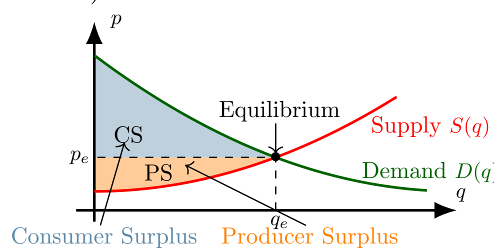


$$\boxed{\int_a^b f(x)\, dx \approx [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]\cdot\frac{\Delta x}{2}} \leftarrow \textbf{trapezoidal rule}$$

▶ **Consumer/Producer Surplus:** In a free market economy, both the consumers and producers benefit by buying/selling at the equilibrium price. It can be shown (see Section 6.1) that:

$$\boxed{\text{Consumer Surplus (CS)} = \int_0^q D(q)dq - p_e q_e}$$

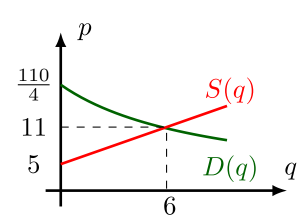
$$\boxed{\text{Producer Surplus (PS)} = p_e q_e - \int_0^{q_e} S(q)dq}$$



Example 5 Find the equilibrium quantity and price, the consumer surplus and the producer surplus for the demand curve $D(q) = \frac{110}{q+4}$ and the supply curve $S(q) = q + 5$. (Ans: qe = 6, pe = 11, CS 34.79, PS = 18)

$D(q) = S(q) \iff \frac{110}{q+4} = q + 5 \iff (q+4)(q+5) = 110 \iff q^2 + 9q + 20 = 110 \iff q^2 + 9q - 90 = 0$

$\iff (q+15)(q-6) = 0 \iff q = -15, 6 \implies q_e = \boxed{6} \implies p_e = S(6) = 6 + 5 = \boxed{11}$



$CS = \int_0^6 \frac{110}{q+4}dq - 6\cdot 11 = 110\ln(q+4)\Big|_0^6 - 66 = 110\ln\left(\frac{10}{4}\right) - 66 \approx \boxed{34.79}$

$PS = 66 - \int_0^6 (q+5)dq = 66 - \left(\frac{1}{2}q^2 + 5q\right)\Big|_0^6 = 66 - \left(\frac{1}{2}\cdot 36 + 30\right) = 66 - 48 = \boxed{18}$