Name ________________________________ Date ____________________

Math 10250 Activity 34: The Fundamental Theorem of Calculus (Sections 5.5, 5.6)

GOAL: Understand the Fundamental Theorem of Calculus (FTC) and use it to compute integrals, including the method of substitution.

Q1: What is the connection between $\int_a^b f(x)\,dx$ (definite integral) and $\int f(x)\,dx$ (indefinite integral)?

A2:

Fundamental Theorem of Calculus (FTC)

IF (1) $f(x)$ is continuous on $[a, b]$ and (2) $F(x)$ is an antiderivative of $f(x)$; i.e., $F'(x) = f(x)$,

THEN $\int_a^b f(x)dx = F(b) - F(a)$;i.e., $\int_a^b F'(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

Example 1 Compute the following definite integrals:

(a) $\int_1^2 (x^2 + 3)\,dx = \frac{x^3}{3} + 3x\Big|_1^2$

$= \frac{2^3}{3} + 3\cdot 2 - \frac{1^3}{3} - 3\cdot 1$

$= \frac{8}{3} + 6 - \frac{1^3}{3} - 3 = \frac{16}{3}$

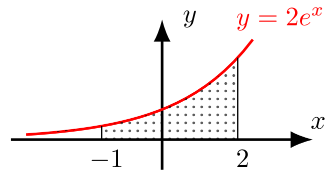
(b) $\int_{-2}^{-1} (e^{2x} + \frac{2}{x})\,dx = \frac{1}{2}e^{2x} + 2\ln|x|\Big|_{-2}^{-1}$

$= \frac{1}{2}e^{2\cdot(-1)} + 2\ln|-1| - \frac{1}{2}e^{2(-2)} - 2\ln|-2|$

$= \frac{1}{2}e^{-2} - \frac{1}{2}e^{-4} - 2\ln 2$

Example 2 Sketch the graph of $f(x) = 2e^x$ from $a = -1$ to $b = 2$ and use the fundamental theorem of calculus to find the area under the graph.

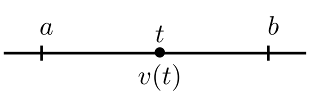
$\int_{-1}^{2} 2e^x\,dx = 2e^x\Big|_{-1}^{2} = 2e^2 - 2e^{-1}$



▶ **Physical interpretations of the Fundamental Theorem of Calculus**

** **Total change** of a certain quantity is expressed as the **definite integral** of its rate of change.**

• **From velocity v to displacement s:**



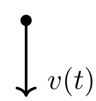
Displacement between times a and b = $s(b) - s(a)$ = $\int_a^b v(t)\,dt$.

↑ change of position between times a and b

Example 3 An object is falling vertically downward, and its velocity (in feet per second) is given by $v = -32t - 20$. Write a definite integral that gives the change in height in the first 3 seconds.

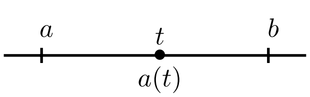
Change in height in the first 3 seconds:

$s(3) - s(0) = \int_0^3 (-32t - 20)dt = (-16t^2 - 20t)\Big|_0^3 = -16(3)^2 - 20(3) = \boxed{-204 \text{ feet}}$



Similary, the following are true.

• **From acceleration a to velocity v:**



Change in velocity between times a and b = $v(b) - v(a)$ = $\int_a^b a(t)\,dt$.

• **From rate of growth $r(t)$ to total growth $g(t)$:**

$$\text{Total growth between times } a \text{ and } b = g(b) - g(a) = \int_a^b r(t)\, dt.$$

▶ **From marginal function to total function**

- The additional profit resulting in increasing production from a units to b units is given by

$$\text{Total change in profit } \stackrel{?}{=} P(b) - P(a) \stackrel{?}{=} \int_a^b P'(x)dx = \int_a^b MP(x)\, dx.$$

- The extra revenue resulting from increasing production from a units to b units is given by

$$\text{Total change in revenue } \stackrel{?}{=} R(b) - R(a) \stackrel{?}{=} \int_a^b R'(x)dx = \int_a^b MR(x)\, dx.$$

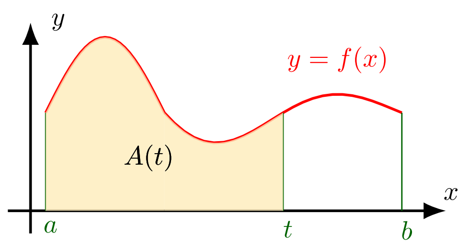
Example 4 Suppose the marginal cost involved in producing x units of a certain product is given by the function

$$MC(x) = 2x + 1000 \text{ when } x \geq 50.$$

Determine the increase in cost if production is increased from 50 to 80.

$$C(80) - C(50) = \int_{50}^{80}(2x + 1000)dx = (x^2 + 1000x)\Big|_{50}^{80} = 80^2 + 1000(80) - [50^2 + 1000(50)] = \boxed{33,900}$$

▶ **The area as an antiderivative**



Let $A(t) = \int_a^t f(x)\, dx$ for $a \leq t \leq b$.

If $F(t)$ is an antiderivative of $f(t)$, what is the relation between $A(t)$ and $F(t)$? (Hint: Fundamental Theorem of Calculus)

<u>Conclusion</u>: $A(t)$ is *also* an antiderivative of $f(t)$, i.e.,

Theorem 5.5.2

IF $f(x)$ is continuous on $[a, b]$ THEN $\dfrac{d}{dt}\displaystyle\int_a^t f(x)dx \stackrel{?}{=} f(t)$

Example 5 $\dfrac{d}{dt}\displaystyle\int_1^t (1 + \ln x)^2\, dx \stackrel{?}{=} (1 + \ln t)^2.$

▶ **Substitution in definite integrals:** $\displaystyle\int_a^b f(g(x))g'(x)\, dx \xlongequal{u=g(x)} \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$

Example 6

(a) $\displaystyle\int_{x=4}^{x=5} x\sqrt{x^2 - 16}\, dx = \int_{u=0}^{u=9} \frac{1}{2}\sqrt{u}du$

$u = x^2 - 16 \implies du = 2xdx \implies xdx = \frac{1}{2}du$
when $x = 4$, *we have*
$u = 4^2 - 16 = 0$
when $x = 5$, *we have*
$u = 5^2 - 16 = 9$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_0^9 = \frac{1}{3}u^{3/2}\Big|_0^9 = \frac{1}{3} \cdot 9^{3/2} = 9$$

(b) $\displaystyle\int_{x=0}^{x=1} xe^{x^2}\, dx = \frac{1}{2}\int_{u=0}^{u=1} e^u du$

$u = x^2 \implies 2xdx = du \implies xdx = \frac{1}{2}du$
when $x = 0$, *we have*
$u = 0^2 = 0$
when $x = 1$, *we have*
$u = 1^2 = 1$

$$= \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e - 1)$$