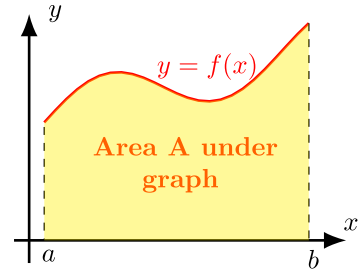
Name ______________________________ Date ____________________

Math 10250 Activity 33: Area and the Definite Integral (Section 5.4)

Goal: To compute the area of **curved** regions in the plane and define the definite integral of "good" functions.

- Consider the region under the graph of a non-negative function $f(x)$ over its domain $[a, b]$:

Q1: How do you compute the area of the region A?



A1: In five steps:
(1) Divide the interval $[a, b]$ into n equal subintervals.

(2) Choose a point in each subinterval.

(3) Compute the area of the rectangle corresponding to each piece.

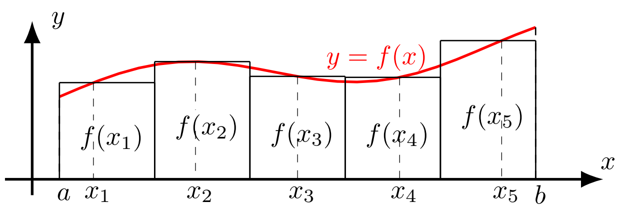
(4) Estimate the area of A by adding the areas of all rectangles.

(5) Get the exact area by taking larger and larger n (smaller and smaller subintervals).

Let's look at each of the above steps in detail.

(1) Divide $[a, b]$ into n subintervals of equal width Δx and choose a point x_i in each subinterval.

(Usually this point is chosen to be either the left endpoint, the right endpoint, or the midpoint of the subinterval.)



$$\boxed{\Delta x = \frac{b-a}{n}} \leftarrow \textbf{mesh of the partition}$$

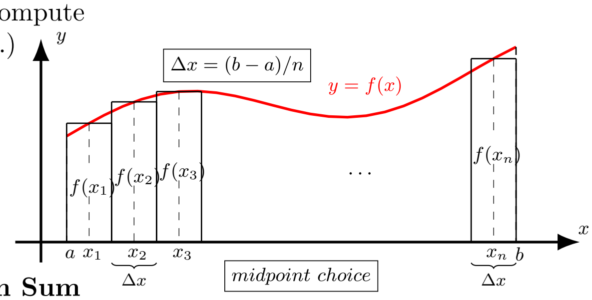
(2) Construct a rectangle over each subinterval with height $f(x_i)$ and compute the area of each rectangle. (Let's use left-hand endpoints of the segments.)

Area of first rectangle = height $\cdot$ base = $f(x_1)\Delta x$.

Area of second rectangle = height $\cdot$ base = $f(x_2)\Delta x$.

$\vdots$

Area of nth rectangle = height $\cdot$ base = $f(x_n)\Delta x$.



(3) Estimate the area of A by adding the areas in (2).

$$\boxed{\text{area of } A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = S_n(f)} \leftarrow \textbf{Riemann Sum}$$

(4) The approximation above gets more accurate as the rectangles get smaller.

(5) So we can get the exact area of A by letting $n \longrightarrow \infty$. Therefore,

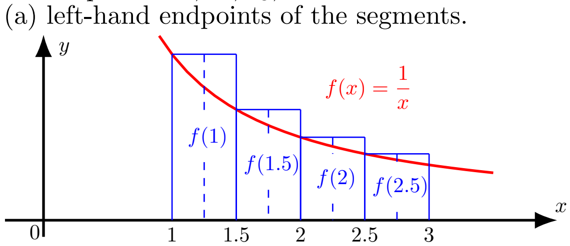
$$\boxed{\text{area of } A = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = \lim_{n\to\infty} S_n(f)}$$

Example 1 Estimate the area under the graph of $y = 1/x$, $1 \le x \le 3$, by partitioning the interval $[1, 3]$ into 4 equal segments and computing the Riemann sum

$$S_4(f) = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x,$$

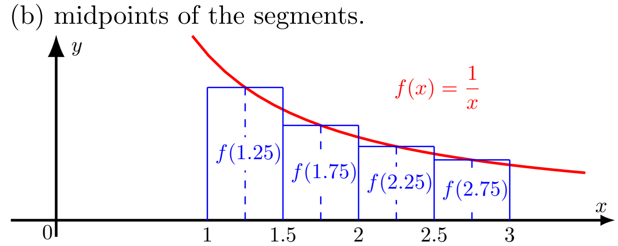
where the points x_1, x_2, x_3, and x_4 are chosen to be:

(a) left-hand endpoints of the segments.



$$S_4 = f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x$$
$$= \frac{1}{1}(0.5) + \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) + \frac{1}{2.5}(0.5) \approx 1.283$$

(b) midpoints of the segments.



$$S_4 = f(1.25)\Delta x + f(1.75)\Delta x + f(2.25)\Delta x + f(2.75)\Delta x$$
$$= \frac{1}{1.25}(0.5) + \frac{1}{1.75}(0.5) + \frac{1}{2.25}(0.5) + \frac{1}{2.75}(0.5) \approx 1.089$$

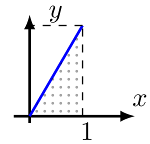
▸ The definite integral: nonnegative case

The limit in Step (5) on the previous page is so special that we give it a name and symbol:

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]} \leftarrow \textbf{Definition of the Definite Integral}$$

↑ area under the graph of $f(x)$ for $a \le x \le b$ if $f(x)$ is nonnegative

If this limit exists, we call it the **Definite Integral of** $f(x)$ **over the interval** $a \le x \le b$.



Example 2 Find $\int_0^1 4x\ dx$ using geometry. $\int_0^1 4xdx =$ *area of triangle* $= \frac{1}{2}\cdot 1\cdot 4 = \boxed{2}$

▸ The Definite integral of a function taking positive and negative values.

We do it in four steps:

(1) Divide the interval $[a, b]$ into n subintervals.

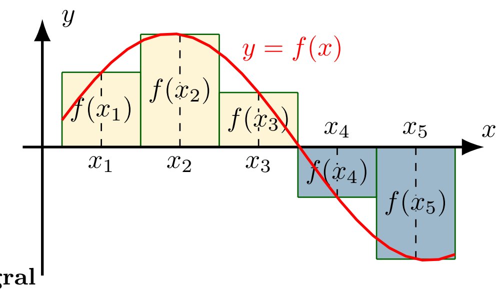
(2) Choose a point in each subinterval.

(3) Compute the corresponding Riemann sum.

$$\boxed{S_n(f) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x} \leftarrow \textbf{Riemann Sum}$$

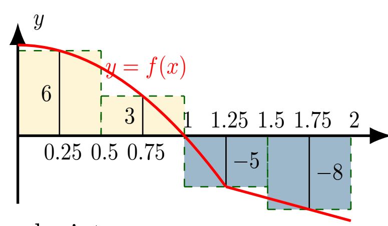
(4) By letting n go to infinity, we obtain:

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]\,.} \leftarrow \textbf{Definite Integral}$$



Example 3 For the function $f(x)$ whose graph is displayed in the figure on the right, estimate $\int_0^2 f(x)dx$ by using the Riemann sum corresponding to $\Delta x = 0.5$ and the midpoints.

$$\int_0^2 f(x)dx \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$
$$= f(0.25)(0.5) + f(0.75)(0.5) + f(1.25)(0.5) + f(1.75)(0.5)$$
$$= 6(0.5) + 3(0.5) + (-5)(0.5) + (-8)(0.5) = \boxed{-2}$$



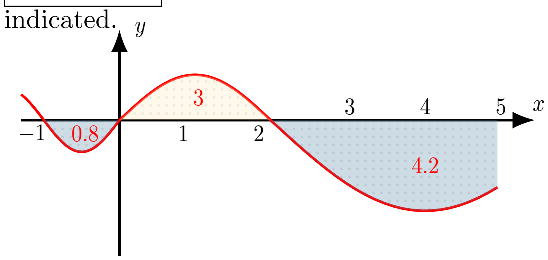
Example 4 Estimate the integral $\int_{-1}^1 x^3 e^{-x^2} dx$ using 4 subintervals and left-hand endpoints.

$$\int_{-1}^1 x^3 e^{-x^2} dx = (-1)^3 e^{-(-1)^2}(0.5) + (-0.5)^3 e^{-(-0.5)^2}(0.5) + (0)^3 e^{-(0)^2}(0.5) + (0.5)^3 e^{-(0.5)^2}(0.5)$$
$$= 0.5[-e^{-1} + (-0.5)^3 e^{-0.5^2} + (0.5)^3 e^{-0.5^2}]$$

- The relation between integral and area is:

$$\boxed{\int_a^b f(x)\ dx = (\text{area of region lying } \textit{over} \text{ the } x\text{-axis}) - (\text{area of region lying } \textit{under} \text{ the } x\text{-axis}).}$$

Example 5 The graph of $f(x)$ for $-1 \le x \le 5$ is shown in the figure below. The size of each enclosed area is as indicated.



$$\int_{-1}^0 f(x)\ dx \overset{?}{=} -0.8 \quad \int_0^2 f(x)\ dx \overset{?}{=} 3 \text{ and } \int_2^5 f(x)\ dx \overset{?}{=} -4.2$$

(a) Find the area of the region **enclosed** by the graph of $f(x)$, $-1 \le x \le 5$, and the x-axis. Area $= 0.8 + 3 + 4.2 = \boxed{5.3}$

(b) $\int_{-1}^5 f(x)dx \overset{?}{=} -0.8 + 3 - 4.2 = \boxed{-2}$

Q2: What are the basic properties of definite integral?

A2: • $\int_a^b cf(x) = c\int_a^b f(x)dx$ • $\int_a^b f(x) + \int_b^c f(x)dx = \int_a^c f(x)dx$

• $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ • $\int_b^a f(x)dx = -\int_a^b f(x)dx$