Name ______________________________ Date __________________

Math 10250 Activity 30: The Indefinite Integral (Section 5.1 continued) and Integration by Substitution (Section 5.2)

GOAL: To learn to solve simple initial value problems and the technique of integration by substitution.

Definition: An equation that involves derivatives of an unknown function is called a **differential equation**. For example, $\dfrac{dy}{dx} = e^{3x}$.

▶ **The initial value problem** (Section 5.1)

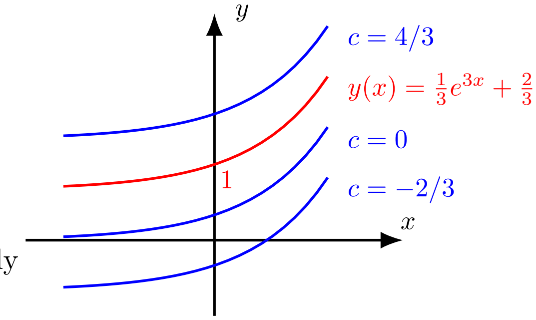
$\boxed{\textbf{Example 1}}$ **(a)** Solve the differential equation $\dfrac{dy}{dx} = e^{3x}$. That is, find the **family** of functions that satisfy the equation $\dfrac{dy}{dx} = e^{3x}$.

$y = \int e^{3x}dx = \frac{1}{3}e^{3x} + c$

(b) Find the unique solution such that $\dfrac{dy}{dx} = e^{3x}$; $\quad y(0) = 1$

$y(0) = \frac{1}{3}e^{3\cdot 0} + c \overset{\text{given}}{=\!=} 1 \Longrightarrow \frac{1}{3} + c = 1 \Longrightarrow c = \frac{2}{3}$

$\Longrightarrow \boxed{y(x) = \dfrac{1}{3}e^{3x} + \dfrac{2}{3}}$

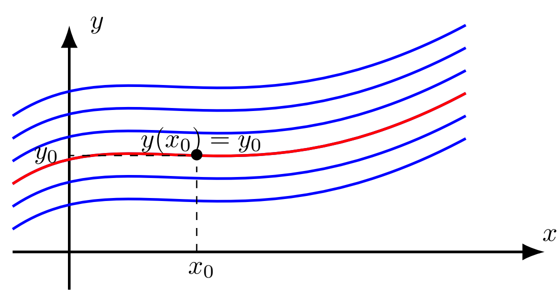


(c) Sketch the solution of (b) and three other members of the family of solutions found in (a) in the axes provided. Label your graphs.

Example 1(b) is an example of an initial value problem. We state its definition below.

Definition: An **initial value problem** consists of a differential equation and an initial condition.

$$\underset{\substack{\uparrow \\ \textbf{differential equation}}}{\frac{dy}{dx} = f(x);} \qquad \underset{\substack{\uparrow \\ \textbf{initial condition}}}{y(x_0) = y_0.}$$

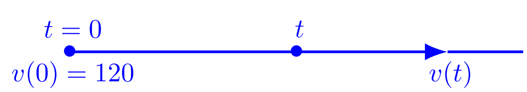


Method for solving initial value problem

Step 1. Find $y(x) = \int f(x)dx = F(x) + c.$

Step 2. Use the initial condition to determine c, i.e. $F(x_0) + c = y_0$.

$\boxed{\textbf{Example 2}}$ A Mustang GT is traveling along a straight road at 120 feet per second when the driver steps on the brake. At that point the car starts decelerating at a constant rate of 20 feet per second squared until it stops. How long does it take to stop?



$v'(t) = -20 \Longrightarrow v(t) = \displaystyle\int (-20)dt = -20t + c$

$120 = v(0) = -20 \cdot 0 + c \Longrightarrow c = 120$

$\Longrightarrow v(t) = -20t + 120$

the car stops when $v = 0$

$0 = -20t + 120 \Longrightarrow t = \frac{120}{20} = 6$ *sec.*

▶ **Integration by substitution** (Section 5.2)

IDEA: Integration "reverses" differentiation. Integration by substitution "reverses" the Chain Rule.

• Let $F(x)$ be an antiderivative of $f(x)$; that is, $F'(x) = f(x)$. Let $g(x)$ be a differentiable function. Using the chain rule:

$$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x) = \quad f(g(x)) \cdot g'(x)$$

So we have: $$\boxed{\int f(g(x))g'(x)dx = F(g(x)) + c, \quad \text{where } F' = f.}$$

The above computation suggests the following formal steps we could take for integration. This method is called **Integration by Substitution**.

• Let $u = g(x)$. Then $\dfrac{du}{dx} = g'(x) \quad \Rightarrow \quad du = g'(x)dx$

• $$\int f(g(x))g'(x)dx \underset{\substack{\uparrow \\ \textbf{u-substitution}}}{=} \int f(u)du = F(u) + c = F(g(x)) + c.$$

$\boxed{\textbf{Example 3}}$ Find the following indefinite integrals by substitution. Check your answer by differentiating.

(a) $$\int (x-5)^{10}dx = \int u^{10}du = \frac{u^{10+1}}{10+1} + c = \frac{1}{11}(x-5)^{11} + c$$

$u = x - 5$
$du = (x-5)'dx = dx$

(b) $$\int \frac{1}{2x-3}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x-3| + c$$

$u = 2x - 3$
$du = 2dx$
$\frac{1}{2}du = dx$

(c) $$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$$

$u = x^2$
$du = 2xdx$
$\frac{1}{2}du = xdx$

(d) $$\int \frac{e^t}{(2+e^t)^2}dt = \int \frac{du}{u^2} = \frac{-u^{-2+1}}{-2+1} + c = -u^{-1} + c = \frac{-1}{2+e^t} + c$$

$u = 2 + e^t$
$du = e^t dt$

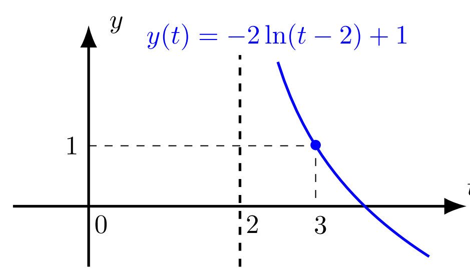
$\boxed{\textbf{Example 4}}$ Solve the initial value problem $\dfrac{dy}{dt} = \dfrac{2}{2-t}; \qquad y(3) = 1.$

$$y(t) = \int \frac{2}{2-t}dt = \int \frac{2}{u}(-du) = -2\ln|u| + c = -2\ln|2-t| + c$$

$u = 2 - t$
$du = -dt$
$-du = dt$

$$1 = y(3) = -2\ln|2-3| + c = -2 \cdot 0 + c$$

$$\Longrightarrow c = 1 \Longrightarrow y(t) = -2\ln|2-t| + 1$$



Since the solution lives near $t = 3$, *we have* $\boxed{y(t) = -2\ln(t-2) + 1,\ t > 2}$.