Name ______________________________ Date ___________________

## Math 10250 Activity 27: Optimization (Section 4.4 continued) and Applied Optimization Problems (Section 4.5)

**GOAL:** To find maximum and minimum of a continuous function over an interval with one or both endpoints excluded.

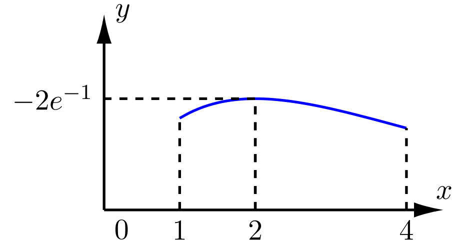
▶ **Case 1: Optimizing $f(x)$ on a closed interval** (Done in last class)

**Example 1** Find the global maximum and minimum of the function $f(x) = xe^{-x/2}$ for $[1, 4]$. Give a sketch of the graph of $f(x)$ clearly indicating where the global maximum and minimum are.

$$f'(x) = 1 \cdot e^{-\frac{x}{2}} + xe^{-\frac{x}{2}}\left(-\frac{1}{2}\right) = \left(1 - \frac{1}{2}x\right)e^{-\frac{x}{2}} \overset{cp}{=} 0$$

$\Longrightarrow$ cp: $x = 2$

| $x$ | 1 | 2 | 4 |
|---|---|---|---|
| $f(x)$ | $e^{-\frac{1}{2}}$ | $2e^{-1}$ | $4e^{-2}$ |
| | | ↑ max | ↑ min |




▶ **Case 2: Optimizing continuous $f(x)$ on an interval with one or both endpoints excluded (i.e., on $(a, b], (-\infty, b], [a, \infty), (-\infty, \infty), \ldots$) - Global maximum and minimum may or may not exist.**

**Example 2** Using the steps below, find the global maximum and minimum of the function $f(x) = xe^{-x/2}$ on $[1, \infty)$.

**Step 1:** Find all critical points in the domain of $f(x)$ and the values of $f(x)$ there. Classify them using the first derivative test.

| $x$ | 1 | | 2 | |
|---|---|---|---|---|
| *sign of* $f'(x)$ | | $+$ | 0 | $-$ |
| *info about* $f(x)$ | | ↗ | max | ↘ |

$$f'(x) = \left(1 - \frac{1}{2}x\right)e^{-\frac{x}{2}}$$

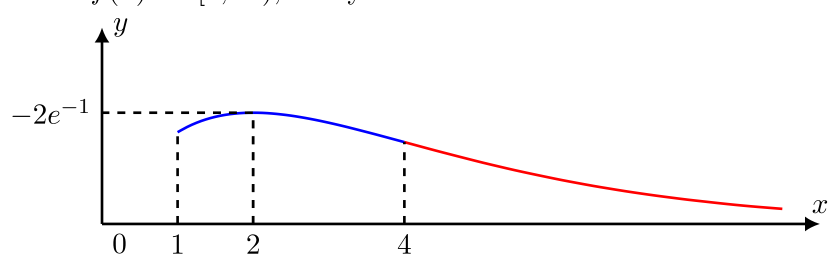
**Step 2:** Find all the asymptotes of $f(x)$ in its domain and determine its asymptotic behavior.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^{\frac{x}{2}}} \approx \frac{\textit{big}}{\textit{Huge}} = 0$$

**Step 3:** Find the values of $f(x)$ at the endpoints (if any) of its domain. $f(1) = e^{-\frac{1}{2}}$

**Step 4:** Give a rough sketch of the graph of $f(x)$ clearly indicating where the global maximum and minimum are. State the global maximum and minimum of $f(x)$ on $[1, \infty)$, if any.

| $x$ | 1 | 2 | $\infty$ |
|---|---|---|---|
| $f(x)$ | $e^{-\frac{1}{2}}$ | $2e^{-1}$ | $\lim_{x\to\infty} f(x) = 0$ |
| | | ↑ max | ↑ min is NOT taken |




**Q1:** How does Example 2 contrast with Example 1?

**A1:** *In Example 2, the right endpoint is $\infty$ and the function $f(x)$ approaches to 0 as $x \longrightarrow \infty$. So, $f(x)$ has no minimum on $[1, \infty)$.*

**Example 3** Find the global maximum and minimum of $f(x) = x^4 - 8x^2$ on $(-\infty, 1)$.

**Step 1:** Find all critical points in the domain of $f(x)$ and the values of $f(x)$ there. Classify them using the first derivative test.

$$f'(x) = 4x^3 - 16x = 4x(x^2-4)$$
$$= 4x(x+2)(x-2) \overset{cp}{=} 0$$
$$\Longrightarrow \boxed{cp: \ x = -2, 0, 2}$$

| $x$ | | $-2$ | | $0$ | | $1$ |
|---|---|---|---|---|---|---|
| *sign of* $f'(x)$ | $-$ | $0$ | $+$ | $0$ | $-$ | |
| *info about* $f(x)$ | $\searrow$ | *min* | $\nearrow$ | *max* | $\searrow$ | |

**Step 2:** Find all the asymptotes of $f(x)$ in its domain and determine its asymptotic behavior.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x^4 - 8x^2) \approx \mathit{Huge} - \mathit{big} = \infty$$

**Step 3:** Find the values of $f(x)$ at the endpoints (if any) of its domain. $f(1) = 1 - 8 = -7$

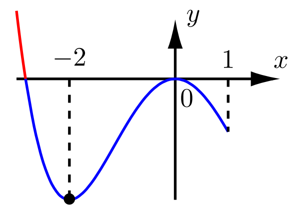
**Step 4:** Give a rough sketch of the graph of $f(x)$ clearly indicating where the global maximum and minimum are. State the global maximum and minimum of $f(x)$ on $(-\infty, 1)$, if any.

$f(-2) = (-2)^4 - 8(2)^2 = 16 - 32 = -16$

*Global min at* $x = -2$ *equals to* $-16$

*No global max*

| $x$ | $-\infty$ | $-2$ | $0$ | $1$ |
|---|---|---|---|---|
| $f(x)$ | $\lim_{x\to-\infty} f(x) = \infty$ | $-16$ | $0$ | $-7$ |
| | $\uparrow$ *max is NOT taken* | $\uparrow$ *min* | | |




**NEXT GOAL:** To use our optimization methods to solve word problems.

**Example 4** A restaurant owner studied the sales of an octopus dish and determined that its average number of orders $q$ each night is given by $p = \dfrac{72}{q+2}$, where $p$ is the price in dollars of an order of the dish. Supposing each appetizer costs the restaurant \$4 to make, help the owner of the restaurant with the following calculations:

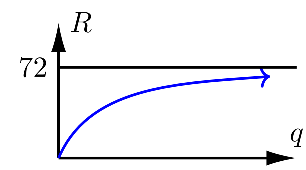
(a) Write down the revenue function: $R = q \cdot p = \dfrac{72q}{q+2}$

(b) What is the largest amount of revenue the restaurant can make from the appetizer?

$$R' = 72\frac{1\cdot(q+2) - q\cdot 1}{(q+2)^2} = \frac{144}{(q+2)^2} > 0$$

$$\lim_{q\to\infty} R(q) = \lim_{q\to\infty}\frac{72}{1+2/q} = 72$$

| $q$ | $0$ | $\infty$ |
|---|---|---|
| $R(q)$ | $0$ | $\lim_{x\to\infty} f(x) = 72$ |
| | $\uparrow$ *min* | $\uparrow$ *max is NOT taken* |




(c) What price should the owner charge to maximize profit from the appetizer?

$$P = R - C = \frac{72q}{q+2} - 4q$$

$$P' = \frac{144}{(q+2)^2} - 4 = 0$$

$$\Longrightarrow (q+2)^2 = 36 \Longrightarrow q + 2 = \pm 6$$

$$\Longrightarrow q = -2 \pm 6 \Longrightarrow \boxed{cp\text{: } q = -8, 4}$$

| $x$ | | $4$ | |
|---|---|---|---|
| *sign of* $P'(x)$ | $+$ | $0$ | $-$ |
| *info about* $P(x)$ | $\nearrow$ | *max* | $\searrow$ |

| $q$ | $0$ | $4$ | $\infty$ |
|---|---|---|---|
| $P(q)$ | $0$ | $32$ | $\lim_{x\to\infty} f(x) = -\infty$ |
| | | $\uparrow$ *max* | $\uparrow$ *min is NOT taken* |

*Global max at* $q = 4$. *Then*

$$P(4) = \frac{72\cdot 4}{4+2} - 4\cdot 4 = \frac{72\cdot 4}{6} - 16 = 32.$$