Name ______________________________ Date ____________________

Math 10250 Activity 26: Optimization on Closed and Bounded Intervals (Section 4.4)

GOAL: To optimize a continuous function $f(x)$ over a closed and bounded interval $[a, b]$. That is, we want to find the global maximum and global minimum of $f(x)$ for $a \le x \le b$.

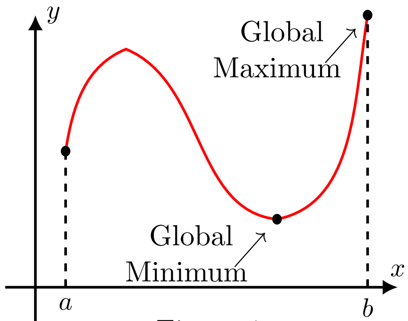


Figure 1

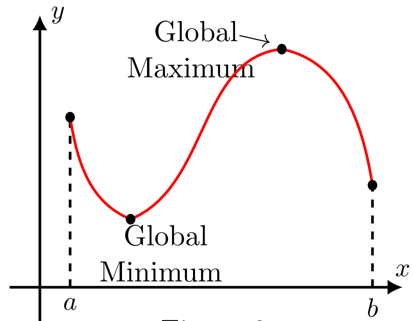


Figure 2

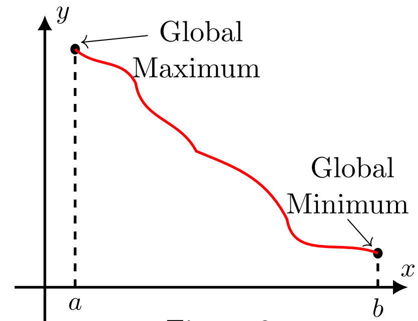


Figure 3

From Figures 1, 2, and 3, we can observe the following fact:

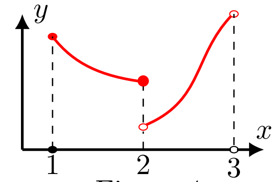


Figure 4

The extreme value theorem
If $f(x)$ is continuous on $[a, b]$, then $f(x)$ has a global maximum and a global minimum on $[a, b]$. (*Like Figure 1-3. In Figure 4, max/min not taken*)

Q1: If $f(x)$ is continuous, where are possible places for the global maximum and global minimum of $f(x)$ to occur on $[a, b]$? (See Figures 1, 2, and 3.)

A1: On a closed and bounded interval $[a, b]$, a continuous function $f(x)$ attains its global maximum and global minimum at (1) *critical points (cp)* or (2) *endpoints*.

Method for finding global maxima and minima of f on $[a, b]$
1. Find all critical points in (a, b).
2. Evaluate f at all critical points and at endpoints. Then compare the values of f:
   highest = global maximum and **lowest** = global minimum.

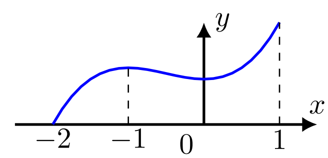
Example 1 Find the global maximum and minimum of the following functions on the given interval:

a. $f(x) = 2x^3 + 3x^2 + 4$ over $[-2, 1]$.

$f'(x) = 6x^2 + 6x = 6x(x+1) \overset{cp}{=} 0$

$\Longrightarrow \boxed{\text{cp: } x = -1, 0}$

x	-2	-1	0	1
$f(x)$	0	5	4	9
	↑ *min*			↑ *max*



b. $f(x) = \dfrac{x}{x^2 + 4}$ over $[0, 3]$.

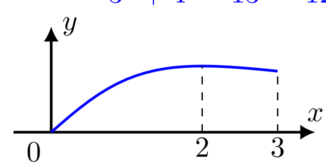
$$f'(x) = \frac{1(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{x^2 + 4 - 2x^2}{(x^2+4)^2}$$
$$= \frac{4 - x^2}{(x^2+4)^2} = \frac{(2-x)(2+x)}{(x^2+4)^2} \overset{cp}{=} 0$$

$\Longrightarrow \boxed{\text{cp: } x = \pm 2}$

Note $x = -2$ is not in the special domain.

x	0	2	3
$f(x)$	0	$\frac{1}{4}$	$\frac{3}{13}$
	↑ *min*	↑ *max*	

$$f(2) = \frac{2}{2^2+4} = \frac{2}{8} = \frac{1}{4}$$
$$f(3) = \frac{3}{3^2+4} = \frac{3}{13} < \frac{3}{12} = \frac{1}{4}$$



Example 2 The demand function for a certain product is given by $p = e^{-q/10}$ where p is the price in dollars and q is the weekly demand in million of units. Assuming that the company can produce no more than 5 million units of the product each week, find the production level that maximizes the revenue of the company. Would your answer change if the company could produce up to 20 million units of the product each week? If so how?

Reveneu : $R(q) = q \cdot p = q \cdot e^{-\frac{1}{10}q}$

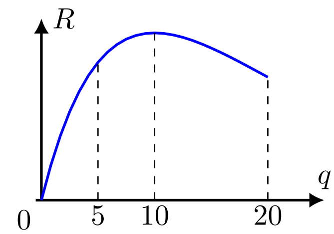
$$R'(q) = 1 \cdot e^{-\frac{1}{10}q} + q \cdot e^{-\frac{1}{10}q} \cdot \left(-\frac{1}{10}\right)$$
$$= \left(1 - \frac{1}{10}q\right) e^{-\frac{1}{10}q} \overset{cp}{=} 0$$
$$\Longrightarrow \boxed{\textit{cp: } q = 10}$$

$R(0) = 0$

$R(5) = 5e^{-\frac{1}{10}\cdot 5} = 5e^{-\frac{1}{2}}$

$R(10) = 10e^{-\frac{1}{10}\cdot 10} = 10e^{-1}$

$\frac{10}{e} > \frac{5}{e^{\frac{1}{2}}} \Longleftrightarrow 2e^{\frac{1}{2}} > e \Longleftrightarrow 2 > \sqrt{e}$

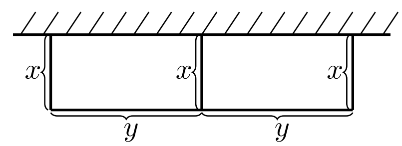


q	0	5	10	20
$R(q)$	0	$5e^{-\frac{1}{2}}$	$10e^{-1}$	$20e^{-2}$

↑ *max over* $[0, 5]$ (at $q = 5$) ↑ *max over* $[0, 20]$ (at $q = 10$)

Example 3 A landscaper plans to use 120m of fencing and a very wide straight wall to make two rectangular enclosures with the same dimensions as shown.

a. Write down the possible values.



Length of fence $= 3x + 2y = 120 \Longrightarrow 2y = 120 - 3x \Longrightarrow y = 60 - \frac{3}{2}x$

x *is the biggest when* $y = 0 \Longrightarrow 3x + 0 = 120 \Longrightarrow x = 40$

b. Find the maximum value of the total area of the enclosures. What are the dimensions of each enclosure when the maximum occurs?

$$\textit{Area} = A = x \cdot y + x \cdot y$$
$$= 2xy = 2x\left(60 - \frac{3}{2}x\right) = 120x - 3x^2$$
$$\Longrightarrow \textit{Need to maximize } A(x) = 120x - 3x^2 \quad \textit{on} \quad [0, 40]$$

$$A'(x) = 120 - 6x = 6(20 - x) \overset{cp}{=} 0 \Longrightarrow \boxed{\textit{cp: } x = 20}$$

$A(0) = 120 \cdot 0 - 3 \cdot 0^2 = 0$

$A(20) = 2 \cdot 20\left(60 - \frac{3}{2} \cdot 20\right) = 40 \cdot 30 = 1200$

$A(40) = 2 \cdot 40\left(60 - \frac{3}{2} \cdot 40\right) = 80 \cdot 0 = 0$

x	0	20	40
$A(x)$	0	1200	0

↑ *max*